1. (18%) Let \( Y_i = \alpha + \beta x_i + \varepsilon_i, \; i = 1, \ldots, n \), where \( \varepsilon_i \) are independently distributed as \( N(0, \sigma^2) \). Find the MLEs of \( \alpha, \beta \) and \( \sigma^2 \) and the distributions of these estimates.

2. (16%) Let the stochastically independent random variables \( X \) and \( Y \) have distributions that are \( N(\theta_1, \theta_3) \) and \( N(\theta_2, \theta_3) \), respectively, where \( \theta_1, \theta_2, \theta_3 \) are unknown. Let \( X_1, \ldots, X_n \) and \( Y_1, \ldots, Y_n \) be two independent random samples from these distributions. Derive the likelihood ratio test for testing \( H_0: \theta_1 = \theta_2 \) versus \( H_1: \theta_1 \neq \theta_2 \).

3. (16%) Let \( X_1, \ldots, X_n \) be a random sample from \( N(\theta, 1) \), where the mean \( \theta \) is unknown. Consider testing the simple hypothesis \( H_0: \theta = \theta_0 \), where \( \theta_0 \) is known, against \( H_1: \theta \neq \theta_0 \). Show that there is no uniformly most powerful test.

4. (10%) Let \( X_1 \) and \( X_2 \) be iid random variables with the pdf \( f(x) = e^{-x}, x \in (0, \infty) \). Show that \( Y_1 = \frac{X_1}{X_1 + X_2} \) and \( Y_2 = X_1 + X_2 \) are independent.

5. (20%) If the independent variables \( Y_1 \) and \( Y_2 \) have means \( \mu_1, \mu_2 \) and variances \( \sigma_1^2 \) and \( \sigma_2^2 \), respectively. Find (a) the mean and variance of the product \( W = Y_1 Y_2 \) and (b) the covariance and correlation of \( Y_1 \) and \( W \) in terms of the means and variances of \( Y_1 \) and \( W \).

6. (20%) Let \( X_{ij}, i = 1, \ldots, m \), and \( X_{ij}, j = 1, \ldots, n \), be two random samples from the Bernoulli \( (p_1) \) and Bernoulli \( (p_2) \) population, respectively. Define \( \hat{p}_1 = \frac{\sum X_{i1}}{m} \) and \( \hat{p}_2 = \frac{\sum X_{j1}}{n} \), and \( \hat{p} = \left\{ \frac{\sum X_{i1} + \sum X_{j1}}{m+n} \right\} \). Describe and compare the asymptotic distributions of \( U_1 \) and \( U_2 \) for (a) \( p_1 = p_2 \), and (b) \( p_1 \neq p_2 \), where

\[
U_1 = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{m} + \hat{p}_2(1-\hat{p}_2)/n}} \quad \text{and} \quad U_2 = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1-\hat{p})/(m+n + 1/n)}}.
\]