Instructions: Read the following essay entitled "The Assignment Problem" first, and then answer the questions following it.

The Assignment Problem

A basic problem in operations research is to assign tasks to facilities on a one-to-one basis in some optimal way. For example, the problem may be to find the best assignment of workers to jobs, sports players to field positions, equipment to construction sites, and so forth. The assignment problem requires that there be as many facilities as tasks, say $n$ of each. In this case, there are exactly $n!$ different ways to assign the tasks to the facilities on a one-to-one basis. This follows because there are $n$ ways to assign the first task, $n - 1$ ways to assign the second, $n - 2$ ways to assign the third, and so on, a total of $n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1 = n!$ possible assignments. Among these $n!$ possible assignments, we are to find one that is optimal in some sense.

To define the notion of an optimal assignment precisely, we introduce the following quantities. Let $c_{ij}$ be the cost of assigning the $i$-th facility the $j$-th task for $i, j = 1, 2, \ldots, n$. The units of $c_{ij}$ might be dollars, miles, hours—whatever is appropriate to the problem. We define the cost matrix to be the $n \times n$ matrix

$$
\begin{bmatrix}
c_{11} & c_{12} & \cdots & c_{1n} \\
c_{21} & c_{22} & \cdots & c_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
c_{n1} & c_{n2} & \cdots & c_{nn}
\end{bmatrix}
$$

The requirement that each facility be assigned a unique task on a one-to-one basis is equivalent to the condition that no two of the corresponding $c_{ij}$'s come from the same row or column. For a given $n \times n$ cost matrix $C$, an assignment is a set of $n$ entry positions, no two of which lie in the same row or column. The sum of the $n$ entries of an assignment is called its cost. An assignment with the smallest possible cost is called an optimal assignment. The assignment problem is to find an optimal assignment in a given cost matrix. For example, in assigning $n$ pieces of equipment to $n$ construction sites, $c_{ij}$ could be the distance in miles between the $i$-th piece of equipment and the $j$-th construction site. An optimal assignment is one for which the total distance traveled by the $n$ pieces of equipment is a minimum. The following theorem leads to a method of converting an arbitrary assignment problem to one that can be solved as easily.

**Theorem:** If a number is added to or subtracted from all of the entries of any one row or column of a cost matrix, then an optimal assignment for the resulting cost matrix is also an optimal assignment for the original cost matrix.
We now introduce the Hungarian method, which is a five step procedure for applying this theorem to a given cost matrix and obtaining one with nonnegative entries that contains an assignment consisting entirely of zero entries. Such an assignment (called an optimal assignment of zero) will then be an optimal assignment for the original problem. The Hungarian method is outlined as following for a given \( n \times n \) cost matrix.

**The Hungarian Method**

1. subtract the smallest entry in each row from all the entries of its row;
2. subtract the smallest entry in each column from all the entries of its column;
3. draw lines through appropriate rows and columns so that all the zero entries of the cost matrix are covered and the minimum number of such lines is used;
4. test for optimality
   (i) if the minimum number of covering lines is \( n \), an optimal assignment of zeros is possible and we are finished;
   (ii) if the minimum number of covering lines is less than \( n \), an optimal assignment of zeros is not yet possible; proceed to step 5;
5. determine the smallest entry not covered by any line. Subtract this entry from all uncovered entries and then add it to all entries covered by both a horizontal and a vertical line; return to step 3.

The first two steps use the above to generate a cost matrix with nonnegative entries and with at least one zero entry in each row and column. The last three steps are applied iteratively as many times as necessary to generate a cost matrix that contains an optimal assignment of zeros. Each time Step 5 is applied, the sum of the entries of the new cost matrix generated is strictly less than the sum of the entries of the preceding cost matrix. This guarantees that the iterative process cannot continue indefinitely. The assignment problems and associated cost matrices that can be solved by the Hungarian methods must satisfy the following three conditions: the cost matrix must be square; the entries of the cost matrix should be integers; and the problem must be one of minimization.
Minimizing Total Distance Traveled - An Example
A construction company has four large bulldozers located at four different garages. The bulldozers are to be moved to four different construction sites. The distances in miles between the bulldozers and construction sites are given below.

<table>
<thead>
<tr>
<th>Construction Sites</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
</tr>
<tr>
<td>1 90</td>
</tr>
<tr>
<td>2 35</td>
</tr>
<tr>
<td>3 125</td>
</tr>
<tr>
<td>4 45</td>
</tr>
</tbody>
</table>

Answer the following Questions based on the above essay:

1. (20%) Explain the following terms: a cost matrix, an assignment, the cost of an assignment, an optimal assignment

2. (10%) Explain what an assignment problem is, and then provide such an example interesting to you.

3. (10%) Does the idea of brute force work for finding an optimal assignment among all possible $n!$ assignments in an assignment problem?

4. (15%) Briefly explain the reason that the iterative processes of the Hungarian method will end after a finite number of iterations.

5. (15%) Explain how to modify the Hungarian method to find a maximum?

6. For the given example, how should the bulldozers be moved to the construction sites in order to minimize the total distance traveled?
   a. (10%) Find the cost matrix for the above problem?
   b. (20%) Apply the Hungarian method to find an optimal solution.