1. (10%) Suppose that \( \lim_{x \to a} f(x) = \infty \) and \( \lim_{x \to a} g(x) = c \), where \( c > 0 \) is a real number.

Prove that \( \lim_{x \to a} [f(x)g(x)] = \infty \) using the arguments with \( \epsilon \) and \( \delta \).

2. (10%) Find the value of \( \lim_{x \to 3} \left( \frac{x}{x - 3} \int_0^x \sin t \frac{1}{t} dt \right) \).

3. (10%) For what values of \( c \) does the polynomial \( P(x) = x^4 + cx^3 + x^2 \) have two inflection points?

One inflection point? None?

4. (a) (5%) Evaluate \( \int_0^n [x] dx \), where \( n \) is a positive integer and \( [x] \) is the greatest integer function (or called as the floor function) that gives the largest integer less than or equal to \( x \).

(b) (5%) Evaluate \( \int_a^b [x] dx \), where \( a \) and \( b \) are real numbers with \( 0 \leq a < b \).

5. (a) (5%) Evaluate \( \int e^{-x^2} dx \) as an infinite series.

(b) (5%) Evaluate \( \int_0^1 e^{-x^2} dx \) correct to within an error of 0.001.

6. (a) (10%) Assume \( A \) is an \( m \) by \( n \) matrix. Prove that the system of equations \( Ax = 0 \) has a non-trivial solution if and only if \( \text{rank}(A) < n \).

(b) (5%) Show that no linearly independent subset of a vector space can contain the zero element.

(c) (10%) Let \( R_2[x] \) be the vector space of all real polynomials of degree at most 2. Consider \( p(x) = 2 + x + x^2 \), \( q(x) = x + 2x^2 \), and \( r(x) = 2 + 2x + 3x^2 \). Prove or disprove that \( p(x) \), \( q(x) \), and \( r(x) \) are linearly independent in \( R_2[x] \). (Hint: You can use the results of part (a) and (b) directly in part (c).)
7. (10%) Solve the following system of equations for all possible real values of $\beta$:

\[
\begin{align*}
  x + y + z + t &= 4, \\
  x + \beta y + z + t &= 4, \\
  x + y + \beta z + (3-\beta)t &= 6, \\
  2x + 2y + 2z + \beta t &= 6.
\end{align*}
\]

8. (15%) Find the eigenvalues and eigenvectors of the matrix $B$,

\[
B = \begin{bmatrix}
  1 & -3 & 3 \\
  3 & -5 & 3 \\
  6 & -6 & 4
\end{bmatrix}
\]

Is this matrix diagonalizable? Why?