1. (a) If $T$ is an estimator of the parameter $\theta$, $C_1$, and $C_2$ are known constants such that $E(T) = \frac{\theta + C_1}{C_2}$, construct an unbiased estimator of $\theta$ and show your claim. (6%) 

(b) Let $T$ be an unbiased estimator of $\theta$. Will $\frac{1}{T}$ be an unbiased estimator of $\frac{1}{\theta}$? Why? (5%) 

(c) Is the maximum likelihood estimate of a parameter always unbiased? Please give two examples. (5%) 

2. (a) What is an unbiased test? (5%) 

(b) Is the likelihood ratio test unbiased? (5%) 

(c) Let $X_1, \ldots, X_n$ be a random sample from the normal distribution $N(\mu, \sigma^2)$. Derive the likelihood ratio test for testing $H_0 : \sigma^2 = \sigma_0^2$ vs. $H_1 : \sigma^2 > \sigma_0^2$. (7%) 

3. Let $X$ be a random variable with the probability density function 

$$f(x) = \begin{cases} 
\frac{1}{\Gamma(r/2)2^{r/2}} x^{r/2-1} e^{-x/2} & , 0 < x < \infty \\
0 & , elsewhere.
\end{cases}$$

(a) Find the moment generating function of $X$. (6%) 

(b) Find $E(X)$ and $Var(X)$ from (a). (6%) 

(c) Find $E\left(\frac{1}{X}\right)$. (5%)
4. Given the following information:

\[ C_t = \hat{\alpha}_1 + 0.92Y_t + e_{1t}, \]
\[ C_t = \hat{\alpha}_2 + 0.84C_{t-1} + e_{2t}, \]
\[ C_{t-1} = \hat{\alpha}_3 + 0.78Y_t + e_{3t}, \]
\[ Y_t = \hat{\alpha}_4 + 0.55C_{t-1} + e_{4t}, \]

where \( e_{1t}, e_{2t}, e_{3t}, \) and \( e_{4t} \) are residuals.

Calculate the least-squares estimates of \( \beta_2 \) and \( \beta_3 \) in

\[ C_t = \beta_1 + \beta_2 Y_t + \beta_3 C_{t-1} + u_t, \]

where \( u_t \) is the disturbance. (15%)

5. The random variable \( X \) has a probability density function \( f(x) \) and cumulative distribution function \( F(x) \).

(a) What is the distribution of the sample maximum? (10%)

(b) Assume the random variable \( X \) has the following probability density function:

\[ f(x) = \frac{1}{\theta}, \quad 0 \leq x \leq \theta. \]

In random sampling from this distribution, prove that the sample maximum is a consistent estimator of \( \theta \). (10%)

6. Suppose that the regression model is

\[ Y_t = \alpha + \beta X_t - \varepsilon_t, \]

\[ f(\varepsilon_t) = \left( \frac{1}{\lambda} \right) \exp[-\varepsilon_t / \lambda], \quad \varepsilon_t \geq 0. \]

Note that the disturbances have \( E[\varepsilon_t] = \lambda \). Show that the least-squares estimates of \( \beta \) is unbiased but the least-squares estimate of \( \alpha \) is biased. (15%)