1. (a) If \( T \) is an estimator of the parameter \( \theta \), \( C_1 \), and \( C_2 \) are known constants such that \( E(T) = \frac{(\theta + C_1)}{C_2} \),

construct an unbiased estimator of \( \theta \) and show your claim. (6%)

(b) Let \( T \) be an unbiased estimator of \( \theta \). Will \( \frac{1}{T} \) be an unbiased estimator of \( \frac{1}{\theta} \)? Why? (5%)

(c) Is the maximum likelihood estimate of a parameter always unbiased? Please give two examples. (5%)

2. (a) What is an unbiased test? (5%)

(b) Is the likelihood ratio test unbiased? (5%)

(c) Let \( X_1, \ldots, X_n \) be a random sample from the normal distribution \( N(\mu, \sigma^2) \).

Derive the likelihood ratio test for testing

\[ H_0 : \sigma^2 = \sigma_0^2 \]

vs.

\[ H_1 : \sigma^2 > \sigma_0^2. \] (6%)

3. A random variable \( X \) is said to be lognormally distributed with parameters \( \mu \) and \( \sigma^2 \)

if \( Y = \log_e X \) is normally distributed with mean \( \mu \) and variance \( \sigma^2 \).

(a) Find the probability density function of \( X \). (5%)

(b) Show that \( E(X^r) = e^{r\mu + r^2\sigma^2/2} \). (6%)

(c) Find \( P(X \leq x) \). (7%)
4. Given the following information:

\[ C_i = \hat{\alpha}_1 + 0.92Y_i + e_{1i}, \]
\[ C_t = \hat{\alpha}_2 + 0.84C_{t-1} + e_{2t}, \]
\[ C_{t-1} = \hat{\alpha}_3 + 0.78Y_t + e_{3t}, \]
\[ Y_t = \hat{\alpha}_4 + 0.55C_{t-1} + e_{4t}, \]

where \( e_{1i}, e_{2t}, e_{3t}, \) and \( e_{4t} \) are residuals.

Calculate the least-squares estimates of \( \beta_2 \) and \( \beta_3 \) in

\[ C_t = \beta_1 + \beta_2 Y_t + \beta_3 C_{t-1} + u_t, \]

where \( u_t \) is the disturbance. (15%) 

5. The random variable \( X \) has a probability density function \( f(x) \) and cumulative distribution function \( F(x) \).

(a) What is the distribution of the sample maximum? (10%) 

(b) Assume the random variable \( X \) has the following probability density function:

\[ f(x) = \frac{1}{\theta}, \quad 0 \leq x \leq \theta. \]

In random sampling from this distribution, prove that the sample maximum is a consistent estimator of \( \theta \). (10%) 

6. Suppose that the regression model is

\[ Y_i = \alpha + \beta X_i - \varepsilon_i, \]
\[ f(\varepsilon_i) = \left( \frac{1}{\lambda} \right) \exp\left[ -\varepsilon_i / \lambda \right], \quad \varepsilon_i \geq 0. \]

Note that the disturbances have \( E[\varepsilon_i] = \lambda \). Show that the least-squares estimates of \( \beta \) is unbiased but the least-squares estimate of \( \alpha \) is biased. (15%)