1. [10%] Let \( D = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \) be an invertible matrix where \( A \) is an \( m \times m \) sub-matrix and \( C \) is an \( n \times n \) sub-matrix. Both \( m \) and \( n \) are positive integers.

1.1. [3%] Suppose that the inverse matrix of \( C \) is known. How to obtain the inverse matrix of the matrix \( E = \begin{bmatrix} C & 0 \\ B & A \end{bmatrix} \) without computations?

1.2. [3%] Express \( \det(E) \) in terms of \( \det(C) \).

1.3. [4%] Assume the linear equation system \( Dx = b \) has a unique solution \([1, 2, 3, 4]^T\). List all the possible solutions for the equation \( Ex = b \).

2. [5%] Solve the unknown \( 3 \times 1 \) vector \( x \) of the linear equation system

\[
\begin{bmatrix}
9 & 0 & 0 \\
2 & 9 & 0 \\
1 & 2 & 9
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} =
\begin{bmatrix}
9 \\
2 \\
1
\end{bmatrix}.
\]

3. [10%] Let \( S \) be the set of all length-\( N \) sequences, i.e., an element \( s \) in the set \( S \) can be written as \( s = (x_0, x_1, \ldots, x_{N-1}) \). Define the addition associated with \( S \) as an entry-by-entry operation, and the multiplication by scalar as \( cs = (cx_0, cx_1, \ldots, cx_{N-1}) \). Then \( S \) can be viewed as equivalent (isomorphic) to the vector space \( \mathbb{R}^N \).

3.1. [2%] Define a decimation transformation \( O_d \) by \( O_d(x_0, x_1, x_2, x_3, x_4, x_5, \ldots) = (x_0, 0, x_2, 0, x_4, 0, \ldots) \). Prove that it is linear.

3.2. [3%] What is the matrix transformation corresponding to \( O_d \) in \( \mathbb{R}^N \)?

3.3. [5%] Define another decimation transformation \( O_{2d}(x_0, x_1, x_2, x_3, x_4, x_5, \ldots) = (x_0, 0, x_3, 0, x_6, 0, \ldots) \). Let \( N = 4 \), and let the corresponding matrix transformations be \( M_d \) and \( M_{2d} \) respectively. Find out the null space and column space of the matrix \( (M_d, M_{2d}) \).

4. [5%] Let \( A \) and \( B \) be square matrices. Let us define that matrix \( A \) is similar to matrix \( B \) if there is an invertible matrix \( P \) such that \( A = P^{-1}BP \). For the four matrices \( A^T, B^T, A^{-1}, B^{-1} \) and the superscript \( ^T \) denotes matrix transposition, answer the following questions.

i. (3) Show that \( A^T \) is similar to \( B^T \).

ii. (2) Show that \( A^{-1} \) is similar to \( B^{-1} \).

5. [20%] Define a quadratic form in the \( n \) variables \( x_1, x_2, \ldots, x_n \) as an expression
that can be written as \[
\begin{bmatrix}
x_1 \\
x_2 \\
m \\
x_n
\end{bmatrix} = x^T A x,
\]
where the matrix \( A \) is a symmetric \( n \times n \) matrix, the vector \( x^T = [x_1, x_2, \ldots, x_n] \) and the superscript \(^T\) denotes matrix transposition. Define a transformation \( T : \mathbb{R}^n \rightarrow \mathbb{R} \) as \( T(x) = x^T A x \). Answer the following questions. You must provide explanations for your answers to get the credits. An answer without explanation will receive zero point.

a. [5\%] For the expression \( x_1^2 + x_2^2 + x_3^2 - 2x_1x_3 \), determine the quadratic form in the three variables \( x_1, x_2, x_3 \) and compute the symmetric \( 3 \times 3 \) matrix \( A \).

b. [5\%] For the expression \( (c_1x_1 + c_2x_2 + c_3x_3)^2 \), determine the quadratic form in the three variables \( x_1, x_2, x_3 \) and compute the symmetric \( 3 \times 3 \) matrix \( A \).

c. [3\%] Compute the expression for \( T(x+y) \) in ONLY three terms where the first two terms are \( T(x) \) and \( T(y) \).

d. [3\%] Compute \( \alpha \) in terms of \( k \) when \( T(kx) = \alpha T(x) \)

e. [4\%] Is the transformation linear?

6. [10\%] It is known that there are 2 defective and 8 non-defective items. If these items are inspected at random, one after another, what is the expected number of items that must be chosen for inspection in order to remove all the defective ones?

7. [10\%] Let sample space \( S \) be the square \( 0 \leq x \leq 1, \ 0 \leq y \leq 1 \) in the plane. Consider the probability densities at all \( (x,y) \) are identical, and let \( A \) be the event of \( A = \{(x,y): 0 \leq x \leq 0.5, \ 0 \leq y \leq 1\} \) and let \( B \) be the event of \( B = \{(x,y): 0 \leq x \leq 1, \ 0 \leq y \leq 0.25\} \).

(a) [5\%] Are \( A \) and \( B \) mutually exclusive events?

(b) [5\%] Are \( A \) and \( B \) independent events?

Prove your answers.
8. [5%] The specification of standard deviation of a 1 Kohm precision resistor is ±10 ohm. Three independent fabrication steps may introduce variation. If the standard deviations of the first and the second steps are 5 ohm and 7 ohm, respectively, what's the maximum allowed standard deviation of the third step such that the final specification can be satisfied.

9. [10%] Given that the joint probability density function (pdf) of random variables $X$ and $Y$ is

$$f_{X,Y}(x,y) = \begin{cases} K & -1 \leq x \leq 0, 0 \leq y \leq 2 \\ 0 & \text{elsewhere} \end{cases}$$

Solve the value of $K$ and the pdf of random variable $Z=X-Y$. Clearly explain and show your derivation steps.

10. [10%] A boy is frequently late in meeting his girl friend by an averaged frequency of two times out of three times. When he is late, his late time span is uniformly distributed from 0 to 15 minutes. Whenever he is late for more than 10 minutes, his girl friend will get upset and walk away without seeing him in that meeting. Assumed that they meet daily and their meeting time is always scheduled at 9:00PM after work and ended at 10:00PM. What is the pdf of their meeting time length? What is their averaged daily meeting time length?

11. [5%] Consider throwing 50 dices all together at a time and observe their total sum. If you perform the experiment 1000 times. After the experiment you plot the observed counts corresponding to all possible sums (from 50 to 300). Give the continuous function that you think best fits the observe data plot. Explain your answer clearly.