1. (25%) A linear homogeneous differential equation with constant coefficients has a solution $x^3 \cos x$.

The differential equation has its order at least
(a) 4 (b) 5 (c) 6 (d) 7 (e) 8.

1-2. Let $y_1(x)$ and $y_2(x)$ be two solutions of

$$\begin{cases}
    y^{(2)}(x) + p(x)y^{(1)}(x) + q(x)y(x) = 0 \\
    y(0) = 1.
\end{cases}$$

Then,
(a) $y_1(x) + y_2(x)$ is also a solution.  (b) $y_1(x) \cdot y_2(x)$ is also a solution.
(c) $\frac{y_1(x)}{y_2(x)}$ is also a solution.  (d) none.

1-3. Which of the following differential equations exists a unique solution?

(a) $\begin{cases}
    y^{(2)}(x) + y(x) = 0 \\
    y(0) = y'(x) = 1
\end{cases}$  
(b) $\begin{cases}
    y^{(2)}(x) + y(x) = 0 \\
    y(0) = y'(0) = 1
\end{cases}$  
(c) $\begin{cases}
    y^{(2)}(x) + y(x) = 0 \\
    y(0) = y'(0) = 1
\end{cases}$  
(d) none.

1-4. Which of the following function sets can be two solutions of differential equation

$$y^{(2)}(x) + p(x)y^{(1)}(x) + q(x)y(x) = 0, \quad x \in (-1, 1)$$

where $p(x)$ and $q(x)$ are two properly designed continuous functions?

(a) $y_1(x) = x, \quad y_2(x) = x^2$
(b) $y_1(x) = x, \quad y_2(x) = \sin x$
(c) $y_1(x) = x, \quad y_2(x) = e^x$
(d) $y_1(x) = 1, \quad y_2(x) = x^2$.

1-5. \[ \mathcal{L}^{-1} \left[ \frac{2s^3}{(s+1)(s+2)(s+3)} \right] = ? \]

(a) $e^{-t} + 16e^{-2t} + 27e^{-3t}$  
(b) $-e^{-t} + 16e^{-2t} - 27e^{-3t}$  
(c) $e^{-t} - 16e^{-2t} + 27e^{-3t}$  
(d) none.

2. (10%) Consider the differential equation with constant coefficients

$$y^{(2)}(x) + a \ y^{(1)}(x) + b \ y(x) = u(x)$$

with some unknown but fixed initial condition.

It is known that $y(x) = \sin x$ when $u(x) = \sin x$.

Please calculate $y(x) =$ ? when $u(x) = \cos x$. 

3. (18%) 以下六小題為有倒扣的是非題。若該敘述正確，請答“是”；否則答
“非”。每題計分規則如下：答對給三分；不作答不給分，但不倒扣；答錯不但
不給分且要倒扣本科成績三分。

3.1. Let \( A \) be an \( n \times n \) real symmetric matrix and \( \lambda \) be a scalar. If there exists a
vector \( x \in \mathbb{R}^n \) such that \( Ax = \lambda x \), then \( \lambda \) is an eigenvalue of \( A^n \), and \( x \) is
an eigenvector of \( A^n \) belonging to \( \lambda^n \), for any positive integer \( m \).

3.2. Let \( A = (a_{ij}) \) be an \( n \times n \) matrix. If \( a_{ij} = i \) for \( i = j \), \( a_{ij} = 0 \) for \( i < j \),
and \( a_{ij} = i - 1 \) for \( i > j \), then \( A \) is diagonalizable.

3.3. Let \( \mathbb{Z} \) denote the set of all integers with addition defined in the usual way and
define the scalar multiplication, denoted by \( \circ \), by \( \alpha \circ k = [\alpha] \cdot k \) for all \( k \in \mathbb{Z} \),
where \( [\alpha] \) denotes the greatest integer less than or equal to \( \alpha \). For example,
\( 1.25 \circ 4 = [1.25] \cdot 4 = 1 \cdot 4 = 4 \). \( \mathbb{Z} \) together with these operations is not a vector
space.

3.4. Let \( A = (a_{ij}) \in \mathbb{R}^{n \times n} \), where \( m < n \). Suppose \( a_{ij} = 0 \) if \( j < i \), and \( a_{ij} \neq 0 \) if
\( i \leq j \). Let \( B = E_1 \cdot E_{k-1} \cdots \cdot E_2 \cdot E_1 \cdot A \), where \( E_1, \ldots, E_k \) denote a sequence of
\( m \times m \) elementary matrices, then the row vectors of \( B \) are linearly independent.

3.5. Let a function \( f : \mathbb{R}^2 \to \mathbb{R} \), and \( f(x) = x_1^2 + x_2^2 \), where \( x = (x_1, x_2)^T \in \mathbb{R}^2 \). Let
\( \hat{x} \in \Omega = \{ x \in \mathbb{R}^2 \mid f(x) = 1 \} \). The tangent space of \( \Omega \) at \( \hat{x} \), denoted by \( T(\hat{x}) \), is
defined by \( T(\hat{x}) = \{ v \in \mathbb{R}^2 \mid y^T \nabla f(\hat{x}) = 0 \} \), where \( \nabla f(\hat{x}) = \left( \frac{\partial f(\hat{x})}{\partial x_1}, \frac{\partial f(\hat{x})}{\partial x_2} \right)^T \).
Then, \( \hat{x} \in T(\hat{x}) \).

3.6. Let \( \{ t_1, t_2, \ldots, t_n \} \) be a basis for an inner product space \( V \). Let \( v \in V \) and
\( v = \sum_{i=1}^{n} a_i t_i \), where \( a_i, i = 1, \ldots, n \), are scalars, then \( a_i = \langle t_i, v \rangle \).
4. (4%) Find all vectors in $\mathbb{R}^4$ that are perpendicular to the three vectors

\[
\begin{bmatrix}
1 \\
1 \\
1 \\
1
\end{bmatrix}, \quad \begin{bmatrix}
1 \\
2 \\
3 \\
4
\end{bmatrix}, \quad \begin{bmatrix}
1 \\
9 \\
9 \\
7
\end{bmatrix}
\]

5. (4%) For which choices of the constant $k$ is the following matrix invertible?

\[
\begin{bmatrix}
1 & 1 & 1 \\
1 & 2 & k \\
1 & 4 & k^2
\end{bmatrix}
\]

6. (4%) If $A=QR$ is a QR factorization, what is the relationship between $A^T A$ and $R^T R$?

7. (5%) Diagonalize the following matrix $A=\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. Here $a$ and $b$ are real numbers and $b$ is nonzero.

8. We draw cards, one at a time, from an ordinary deck of 52 cards with replacement. What is the probability that an ace (A) appears before a face (J, Q, K) card? (12%)

9. A random number is select form $(0, \pi/2)$. What is the probability that its sine is greater than its cosine? (6%)

10. Suppose that a soldier's height is normally distributed with mean 170 cm and standard deviation 10 cm, respectively. What is the probability that of 20 randomly selected soldiers, 5 have a height of at least 175 cm? (12%

(Note: $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0.5} e^{-\frac{t^2}{2}} dt = 0.6915$)