PART A. Discrete Mathematics

1. (5 points) Let \( N \) be the set of all natural numbers. Let \( M \) be the set of all finite subsets of \( N \). Is there a one-to-one, onto mapping from \( N \) to \( M \)? Why or why not?

2. (5 points) A natural number \( n \) may be written as \( p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k} \). Calculate the product of all the divisors of a number \( n \).

3. (5 points) Define \( \pi(i) \) as the sum of all digits of the natural number \( i \). For instance, \( \pi(257) = 2 + 5 + 7 = 14 \). Define

\[
\theta(i) = \begin{cases} 
    i & \text{if } i < 10 \\
    \theta(\pi(i)) & \text{otherwise}
\end{cases}
\]

For instance, \( \theta(257) = \theta(\pi(257)) = \theta(14) = \theta(\pi(14)) = \theta(5) = 5 \). Compute \( \theta[\sum_{i=0}^{999} \theta(i)] \).

4. (5 points) Let \( a_1, a_2, \ldots, a_n \) denote a sequence \( L \) of \( n \) integers. Show that there must be a consecutive subsequence of \( L \) whose sum is divisible by \( n \). Note that a consecutive subsequence has the form \( a_j, a_{j+1}, \ldots, a_{j+k} \).

5. (5 points) Is the following claim true? Why or why not?

There is a Hamiltonian path in an \( n \)-dimensional cube, for \( n \in N \).

For problems 6–11, give your answers directly. No computation is necessary.
6. For each of these relations on the set \{1, 2, 3, 4\} decide whether it is reflexive, whether it is symmetric, whether it is antisymmetric, and whether it is transitive?
   a) \{(2,2),(2,3),(2,4),(3,2),(3,3),(3,4)\} (1%)
   b) \{((1,1),(1,2),(2,1),(2,2),(3,3),(4,4)\} (1%)
   c) \{(2,4),(4,2)\} (1%)
   d) \{(1,2),(2,3),(3,4)\} (1%)
   e) \{(1,1),(2,2),(3,3),(4,4)\} (1%)

7. An employee joined a company in 1999 with a starting salary of $50,000. Every year this employ receives a raise of $1000 plus 5% of the salary of the previous year.
   a) Set up a recurrence relation for the salary of this employ n year after 1999. (2%)
   b) What will be the salary of this employ be in 2007? (1%)
   c) Find an explicit formula for the salary of this employ n year after 1999. (2%)

8. Let G be a simple graph with the smallest number of vertices such that 1) the total degrees of G are exactly 240 and 2) all vertices of G have the same degree. How many vertices G has? (3%)

9. How many divisions are required to find gcd(210, 111) using the Euclidean algorithm? (3%)

10. Consider the graph G whose adjacency matrix is

   \[
   \begin{pmatrix}
   0 & 1 & 1 & 1 \\
   1 & 0 & 1 & 1 \\
   1 & 1 & 0 & 0 \\
   1 & 1 & 0 & 0 \\
   \end{pmatrix}
   \]

   a) How many spanning trees has in the graph G? (3%)
   b) Draw one of its spanning trees. (1%)

11. Let H be the graph shown below where the edges represent the streets in a city. A police officer wants to make a trip to patrol each street exactly once. Can such a trip be designed and give your reason? (The police officer may start at any vertex in the graph). (5%)
PART B.

Problems for Probability

Problem 1 (5%)
In a state lottery, a player must choose 8 of the numbers from 1 to 40. The lottery commission then performs an experiment that selects 8 of these 40 numbers. Assuming that the choice of the lottery commission is equally likely to be any of the $\binom{40}{8}$ combinations, what is the probability that a player has at least 6 of the number selected?

Problem 2 (5%)
Suppose we have 10 coins such that if the $i$th coin is flipped, heads will appear with probability $\frac{i}{10}$, $i = 1, 2, \ldots, 10$. When one of the coins is randomly selected and flipped, it shows heads. What is the conditional probability that it was the fifth coin?

Problem 3 (5%)
If the distribution function of a random variable $X$ is given by

$$F(t) = \begin{cases} 0 & t < 0 \\ 1/2 & 0 \leq t < 1 \\ 3/5 & 1 \leq t < 2 \\ 4/5 & 2 \leq t < 3 \\ 9/10 & 3 \leq t < 4 \\ 1 & t \geq 4 \end{cases}$$

calculate the probability mass function of $X$.

Problem 4 (10%)
A sample of 3 items is selected at random from a box containing 20 items of which 4 are defective. Find the expected number of defective items in the sample.

Problem 5 (6%)
Let $X$ be a continuous random variable with range $[0, 1]$ and mean $E(X) = \mu$. Show the inequality $P(X \geq \varepsilon \mu) \geq (1 - \varepsilon)\mu$ for any $0 \leq \varepsilon \leq 1$. 
Problem 6 (6%) 
A student measures the speed of a running train 20 times and take the average of the measurements as the result. If each measurement has an independent random error with standard the standard normal distribution $N(0, 1)$ (in kilometers per hour). What is the probability that the result differs from the actual speed by less than 0.1 kilometers per hour?

Problem 7 (13%; 7% for part a and 6% for part b) 
Let $X$ have a uniform distribution over $(-5, 5)$ and $Y$ have a uniform distribution over $(0, 20)$. Assume that $X$ and $Y$ are independent.

a. Compute the distribution function of $Z = \max(3X, 2Y)$.

b. Compute $E(Z)$ and $\text{Var}(Z)$. 