A. LINEAR ALGEBRA

1. True or False (10 points)
   a) The set of all vectors in a plane not parallel to a fixed vector under the usual
      operations of vector addition and scalar multiplication is a vector space.
   b) If two matrices have the same column space, row space, and null space then
      one is a multiple of the other.
   c) \[
      \begin{vmatrix}
      A & B \\
      B & A \\
      \end{vmatrix} = |A + B| \cdot |A - B| \]
      where \( A \) and \( B \) are two matrices of order \( n \).
   d) The vectors
      \[
      A_1 = (1, -2, -2, 1), \quad A_2 = (0, 1, 1, 0), \quad A_3 = (1, -1, -1, 1)
      \]
      and
      \[
      B_1 = (0, -1, -1, 0), \quad B_2 = (1, 0, 0, 1)
      \]
      span the same linear subspace of \( \mathbb{R}^4 \).
   e) Let \( V \) be a 3-dimensional subspace of \( \mathbb{R}^4 \), then every basis of \( V \) can be
      extended to a basis of \( \mathbb{R}^4 \) by adding one more vector.

2. For each problem below, give your answer directly. (15 points)
   No explanation or computation is required.
   a) Find the largest possible number of independent vectors from the following
      vector set.
      \[
      v_1 = \begin{bmatrix}
      0 \\
      1 \\
      -1 \\
      0
      \end{bmatrix}, \quad v_2 = \begin{bmatrix}
      0 \\
      1 \\
      0 \\
      -1
      \end{bmatrix}, \quad v_3 = \begin{bmatrix}
      1 \\
      0 \\
      0 \\
      -1
      \end{bmatrix}, \quad v_4 = \begin{bmatrix}
      0 \\
      0 \\
      1 \\
      1
      \end{bmatrix}, \quad v_5 = \begin{bmatrix}
      1 \\
      0 \\
      -1 \\
      0
      \end{bmatrix}
      \]
   b) Under what condition on the numbers \( p, q, \) and \( r \) will the following system
      of linear equations be consistent?
      \[
      -2x + y + z = p \\
      x - 2y + z = q \\
      x + y - 2z = r
      \]
   c) Let \( A = LU \) be the LU-factorization of \( A =
      \begin{bmatrix}
      1 & -1 & 0 & 0 \\
      -1 & 2 & -1 & 0 \\
      0 & -1 & 2 & -1 \\
      0 & 0 & -1 & 2
      \end{bmatrix} \)
      Find \( L^{-1} \).
d) Find the complete solution in the form \( x_p + x_n \) to the nonhomogeneous system

\[
x + y + z = 5
\]

\( x_p \) is a particular solution of the nonhomogeneous system and \( x_n \) is a solution to the corresponding homogeneous system.

e) Let

\[
A_1 = (1, 0, 0, 0), \quad A_2 = (0, 1, 0, 0), \quad A_3 = (0, 0, 1, 0), \quad A_4 = (0, 0, 0, 1)
\]

and

\[
B_1 = (2, 1, -1, 1), \quad B_2 = (0, 3, 1, 0), \quad B_3 = (5, 3, 2, 1), \quad B_4 = (6, 6, 1, 3)
\]

be two bases for \( \mathbb{R}^4 \).

Find a nonzero vector which has the same coordinates with respect to the above two bases.

以下各計算題，請清楚標出你最後的答案。 (加底線或加框)

3. (1) In \( \mathbb{R}^3 \), consider the subspace \( S \), where

\[
S = \text{span} \{ (1, 1, 0), (1, 1, 1) \}.
\]

Find the orthogonal projection of \( (1, 0, 0) \) onto the subspace \( S \).

(2) In \( \mathbb{R}^3 \), find the rotation matrix that rotates an angle \( \theta \) about \( y \)-axis counterclockwise.

(3) In \( \mathbb{R}^3 \), \( B_1 = \{ (1, 1, 1), (1, -1, 1), (0, 0, 1) \} \), \( B_2 = \{ (2, 2, 0), (0, 1, 1), (1, 0, 1) \} \)

What is the transition matrix from \( B_1 \) to \( B_2 \)?

You do not need to simplify the answer or to find the inverse matrix.

(4) Let \( T \) be a linear transformation from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \) given by

\[
T(x, y) = (x + 3y, 2x).
\]

Find \( T^n(x, y) \), where \( T^n(x, y) = T(T(...T(T(x, y))...)) \), \( n \) times.

4. For each of the following two cases, are the following two matrices similar?

You must explain your reason.

(1) \[
A = \begin{bmatrix} -2 & 7 \\ -3 & 7 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}
\]

(2) \[
C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}
\]
5. Let $A$ and $B$ be two matrices. Of the following five statements, which ones are correct and which ones are incorrect. (No partial credit for partial answer)

(1) The row space of $AB$ is a subspace of the row space of $A$.
(2) The column space of $AB$ is a subspace of the column space of $A$.
(3) The row space of $A$ is a subspace of the row space of $AB$.
(4) rank$(AB) \geq$ rank$(A)$.
(5) rank$(AB) \leq$ rank$(B)$.  

6. In $\mathbb{R}^3$, consider the inner product $\langle u, v \rangle = au_1 v_1 + bu_2 v_2 + cu_3 v_3$.

where $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$.

(1) State all possible values for which $a$, $b$, and $c$ can be, such that $\langle u, v \rangle$ is an inner product.

(2) Write the Cauchy-Schwarz inequality for this particular inner product.

(3) In the subspace $S$=span $\{(1, 1, 1)\}$, find the vector $v$ which is closest to $(1, 2, 3)$.  


6. Problems for Probability

Problem 1 (5%)
In a state lottery, a player must choose 8 of the numbers from 1 to 40. The lottery commission then performs an experiment that selects 8 of these 40 numbers. Assuming that the choice of the lottery commission is equally likely to be any of the \( \binom{40}{8} \) combinations, what is the probability that a player has at least 6 of the number selected?

Problem 2 (5%)
Suppose we have 10 coins such that if the \( i \)th coin is flipped, heads will appear with probability \( \frac{i}{10}, i = 1, 2, \ldots, 10 \). When one of the coins is randomly selected and flipped, it shows heads. What is the conditional probability that it was the fifth coin?

Problem 3 (5%)
If the distribution function of a random variable \( X \) is given by

\[
F(t) = \begin{cases} 
0 & t < 0 \\
\frac{1}{2} & 0 \leq t < 1 \\
\frac{3}{5} & 1 \leq t < 2 \\
\frac{4}{5} & 2 \leq t < 3 \\
\frac{9}{10} & 3 \leq t < 4 \\
1 & t \geq 4
\end{cases}
\]

calculate the probability mass function of \( X \).

Problem 4 (10%)
A sample of 3 items is selected at random from a box containing 20 items of which 4 are defective. Find the expected number of defective items in the sample.

Problem 5 (6%)
Let \( X \) be a continuous random variable with range \([0, 1]\) and mean \( E(X) = \mu \). Show the inequality \( P(X \geq \varepsilon \mu) \geq (1 - \varepsilon)\mu \) for any \( 0 \leq \varepsilon \leq 1 \).
Problem 6 (6%)
A student measures the speed of a running train 20 times and take the average of the measurements as the result. If each measurement has an independent random error with standard the standard normal distribution $N(0, 1)$ (in kilometers per hour). What is the probability that the result differs from the actual speed by less than 0.1 kilometers per hour?

Problem 7 (13%; 7% for part a and 6% for part b)
Let $X$ have a uniform distribution over $(-5, 5)$ and $Y$ have a uniform distribution over $(0, 20)$. Assume that $X$ and $Y$ are independent.
a. Compute the distribution function of $Z = \max (3X, 2Y)$.
b. Compute $E(Z)$ and $\text{Var}(Z)$.