A. Discrete Mathematics

1. (5 points) Let $N$ be the set of all natural numbers. Let $M$ be the set of all finite subsets of $N$. Is there a one-to-one, onto mapping from $N$ to $M$? Why or why not?

2. (5 points) A natural number $n$ may be written as $p_1^{a_1}p_2^{a_2} \ldots p_k^{a_k}$. Calculate the product of all the divisors of a number $n$.

3. (5 points) Define $\pi(i)$ as the sum of all digits of the natural number $i$. For instance, $\pi(257) = 2 + 5 + 7 = 14$. Define

$$\theta(i) = \begin{cases} i & \text{if } i < 10 \\ \theta(\pi(i)) & \text{otherwise} \end{cases}$$

For instance, $\theta(257) = \theta(\pi(257)) = \theta(14) = \theta(\pi(14)) = \theta(5) = 5$. Compute $\theta(\sum_{i=0}^{999} \theta(i))$.

4. (5 points) Let $a_1, a_2, \ldots, a_n$ denote a sequence $L$ of $n$ integers. Show that there must be a consecutive subsequence of $L$ whose sum is divisible by $n$. Note that a consecutive subsequence has the form $a_j, a_{j+1}, \ldots, a_{j+k}$.

5. (5 points) Is the following claim true? Why or why not?

*There is a Hamiltonian path in an $n$-dimensional cube, for $n \in \mathbb{N}$.*

For problems 6~11, give your answers directly. No computation is necessary.
6. For each of these relations on the set \{1, 2, 3, 4\} decide whether it is reflexive, whether it is symmetric, whether it is antisymmetric, and whether it is transitive?
   a) \{(2,2),(2,3),(2,4),(3,2),(3,3),(3,4)\} (1%) 
   b) \{(1,1),(1,2),(2,1),(2,2),(3,3),(4,4)\} (1%) 
   c) \{(2,4),(4,2)\} (1%) 
   d) \{(1,2),(2,3),(3,4)\} (1%) 
   e) \{(1,1),(2,2),(3,3),(4,4)\} (1%)

7. An employee joined a company in 1999 with a starting salary of $50,000. Every year this employ receives a raise of $1000 plus 5% of the salary of the previous year.
   a) Set up a recurrence relation for the salary of this employ n year after 1999. (2%) 
   b) What will be the salary of this employ be in 2007? (1%) 
   c) Find an explicit formula for the salary of this employ n year after 1999. (2%)

8. Let G be a simple graph with the smallest number of vertices such that 1) the total degrees of G are exactly 240 and 2) all vertices of G have the same degree. How many vertices G has? (3%) 

9. How many divisions are required to find gcd(210, 111) using the Euclidean algorithm? (3%) 

10. Consider the graph G whose adjacency matrix is 

    \[
    \begin{pmatrix}
    0 & 1 & 1 & 1 \\
    1 & 0 & 1 & 1 \\
    1 & 1 & 0 & 0 \\
    1 & 1 & 0 & 0 
    \end{pmatrix}
    
    \]

   a) How many spanning trees has in the graph G? (3%) 
   b) Draw one of its spanning trees. (1%) 

11. Let H be the graph shown below where the edges represent the streets in a city. A police officer wants to make a trip to patrol each street exactly once. Can such a trip be designed and give your reason? (The police officer may start at any vertex in the graph). (5%)
B. Problems for Probability

Problem 1 (5%)
In a state lottery, a player must choose 8 of the numbers from 1 to 40. The lottery commission then performs an experiment that selects 8 of these 40 numbers. Assuming that the choice of the lottery commission is equally likely to be any of the \( \binom{40}{8} \) combinations, what is the probability that a player has at least 6 of the number selected?

Problem 2 (5%)
Suppose we have 10 coins such that if the \( i \)th coin is flipped, heads will appear with probability \( \frac{i}{10} \), \( i = 1, 2, ..., 10 \). When one of the coins is randomly selected and flipped, it shows heads. What is the conditional probability that it was the fifth coin?

Problem 3 (5%)
If the distribution function of a random variable \( X \) is given by

\[
F(t) = \begin{cases} 
0 & t < 0 \\
1/2 & 0 \leq t < 1 \\
3/5 & 1 \leq t < 2 \\
4/5 & 2 \leq t < 3 \\
9/10 & 3 \leq t < 4 \\
1 & t \geq 4 
\end{cases}
\]

Calculate the probability mass function of \( X \).

Problem 4 (10%)
A sample of 3 items is selected at random from a box containing 20 items of which 4 are defective. Find the expected number of defective items in the sample.

Problem 5 (6%)
Let \( X \) be a continuous random variable with range \([0, 1]\) and mean \( E(X) = \mu \). Show the inequality \( P(X \geq \varepsilon \mu) \geq (1 - \varepsilon)\mu \) for any \( 0 \leq \varepsilon \leq 1 \).
Problem 6 (6%) 
A student measures the speed of a running train 20 times and take the average of the measurements as the result. If each measurement has an independent random error with standard the standard normal distribution N(0, 1) (in kilometers per hour). What is the probability that the result differs from the actual speed by less than 0.1 kilometers per hour?

Problem 7 (13%; 7% for part a and 6% for part b) 
Let X have a uniform distribution over (−5, 5) and Y have a uniform distribution over (0, 20). Assume that X and Y are independent.
a. Compute the distribution function of $Z = \max (3X, 2Y)$.
b. Compute E(Z) and Var(Z).