A. LINEAR ALGEBRA

1. True or False (10 points)
   a) The set of all vectors in a plane not parallel to a fixed vector under the usual operations of vector addition and scalar multiplication is a vector space.
   b) If two matrices have the same column space, row space, and null space then one is a multiple of the other.
   c) \[
   \begin{vmatrix} A & B \\ B & A \end{vmatrix} = |A + B| \cdot |A - B| \]
   where \( A \) and \( B \) are two matrices of order \( n \).
   d) The vectors
   \[
   A_1 = (1, -2, -2, 1), \quad A_2 = (0, 1, 1, 0), \quad A_3 = (1, -1, -1, 1)
   \]
   and
   \[
   B_1 = (0, -1, -1, 0), \quad B_2 = (1, 0, 0, 1)
   \]
   span the same linear subspace of \( \mathbb{R}^4 \).
   e) Let \( V \) be a 3-dimensional subspace of \( \mathbb{R}^4 \), then every basis of \( V \) can be extended to a basis of \( \mathbb{R}^4 \) by adding one more vector.

2. For each problem below, give your answer directly. (15 points)
   No explanation or computation is required.
   a) Find the largest possible number of independent vectors from the following vector set.
      \[
      \begin{align*}
      v_1 &= \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \\
      v_2 &= \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \\
      v_3 &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\
      v_4 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
      v_5 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
      v_6 &= \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}
      \end{align*}
      \]
   b) Under what condition on the numbers \( p, q, \) and \( r \) will the following system of linear equations be consistent?
      \[
      \begin{align*}
      -2x + y + z &= p \\
      x - 2y + z &= q \\
      x + y - 2z &= r
      \end{align*}
      \]
   c) Let \( A = LU \) be the LU-factorization of \( A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \).

Find \( L^{-1} \).
d) Find the complete solution in the form $x_p + x_h$ to the nonhomogeneous system

\[
x + y + z = 5
\]

($x_p$ is a particular solution of the nonhomogeneous system and $x_h$ is a solution to the corresponding homogeneous system.)

e) Let

\[
A_1 = (1, 0, 0, 0), \quad A_2 = (0, 1, 0, 0), \quad A_3 = (0, 0, 1, 0), \quad A_4 = (0, 0, 0, 1)
\]

and

\[
B_1 = (2, 1, -1, 1), \quad B_2 = (0, 3, 1, 0), \quad B_3 = (5, 3, 2, 1), \quad B_4 = (6, 6, 1, 3)
\]

be two bases for $\mathbb{R}^3$.

Find a nonzero vector which has the same coordinates with respect to the above two bases.

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3. (1) In $\mathbb{R}^3$, consider the subspace $S$, where

\[
S = \text{span} \{(1, 1, 0), (1, 1, 1)\}.
\]

Find the orthogonal projection of $(1, 0, 0)$ onto the subspace $S$.

(2) In $\mathbb{R}^3$, find the rotation matrix that rotates an angle $\theta$ about y-axis counterclockwise.

(3) In $\mathbb{R}^2$, $B_1 = \{(1, 1, 1), (1, -1, 1), (0, 0, 1)\}$, $B_2 = \{(2, 2, 0), (0, 1, 1), (1, 0, 1)\}$

What is the transition matrix from $B_1$ to $B_2$?

You do not need to simplify the answer or to find the inverse matrix.

(4) Let $T$ be a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ given by

\[
T(x, y) = (x+3y, 2x).
\]

Find $T^n(x, y)$, where $T^n(x, y) = T(T(...T(T(x, y))...))$, $n$ times.

4. For each of the following two cases, are the following two matrices similar? You must explain your reason.

4%
5. Let $A$ and $B$ be two matrices. Of the following five statements, which ones are correct and which ones are incorrect. (No partial credit for partial answer)
   (1) The row space of $AB$ is a subspace of the row space of $A$.
   (2) The column space of $AB$ is a subspace of the column space of $A$.
   (3) The row space of $A$ is a subspace of the row space of $AB$.
   (4) $\text{rank}(AB) \geq \text{rank}(A)$.
   (5) $\text{rank}(AB) \leq \text{rank}(B)$.  

6. In $\mathbb{R}^3$, consider the inner product $\langle u, v \rangle = au_1 v_1 + bu_2 v_2 + cu_3 v_3$ where $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$.
   (1) State all possible values for which $a$, $b$, and $c$ can be, such that $\langle u, v \rangle$ is an inner product.
   (2) Write the Cauchy-Schwarz inequality for this particular inner product.
   (3) In the subspace $S = \text{span} \{ (1, 1, 1) \}$, find the vector $v$ which is closest to $(1, 2, 3)$.  

B. Discrete Mathematics

1. (5 points) Let $N$ be the set of all natural numbers. Let $M$ be the set of all finite subsets of $N$. Is there a one-to-one, onto mapping from $N$ to $M$? Why or why not?

2. (5 points) A natural number $n$ may be written as $p_1^{a_1}p_2^{a_2}\ldots p_k^{a_k}$. Calculate the product of all the divisors of a number $n$.

3. (5 points) Define $\pi(i)$ as the sum of all digits of the natural number $i$. For instance, $\pi(257) = 2 + 5 + 7 = 14$. Define

$$\theta(i) = \begin{cases} 
  i & \text{if } i < 10 \\
  \theta(\pi(i)) & \text{otherwise}
\end{cases}$$

For instance, $\theta(257) = \theta(\pi(257)) = \theta(14) = \theta(\pi(14)) = \theta(5) = 5$. Compute $\theta[\sum_{i=0}^{999} \theta(i)]$.

4. (5 points) Let $a_1, a_2, \ldots, a_n$ denote a sequence $L$ of $n$ integers. Show that there must be a consecutive subsequence of $L$ whose sum is divisible by $n$. Note that a consecutive subsequence has the form $a_j, a_{j+1}, \ldots, a_{j+k}$.

5. (5 points) Is the following claim true? Why or why not?

There is a Hamiltonian path in an $n$-dimensional cube, for $n \in N$.

For problems 6–11, give your answers directly. No computation is necessary.
6. For each of these relations on the set \{1, 2, 3, 4\} decide whether it is reflexive, whether it is symmetric, whether it is antisymmetric, and whether it is transitive?
   a) \{(2,2),(2,3),(2,4),(3,2),(3,3),(3,4)\} (1%)
   b) \{((1,1),(1,2),(1,3),(1,4)\} (1%)
   c) \{(2,4),(4,2)\} (1%)
   d) \{(1,2),(2,3),(3,4)\} (1%)
   e) \{(1,1),(2,2),(3,3),(4,4)\} (1%)

7. An employee joined a company in 1999 with a starting salary of $50,000. Every year this employ receives a raise of $1000 plus 5% of the salary of the previous year.
   a) Set up a recurrence relation for the salary of this employ n year after 1999. (2%)
   b) What will be the salary of this employ be in 2007? (1%)
   c) Find an explicit formula for the salary of this employ n year after 1999. (2%)

8. Let G be a simple graph with the smallest number of vertices such that 1) the total degrees of G are exactly 240 and 2) all vertices of G have the same degree. How many vertices G has? (3%)

9. How many divisions are required to find gcd(210, 111) using the Euclidean algorithm? (3%)

10. Consider the graph G whose adjacency matrix is

\[
\begin{pmatrix}
0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0
\end{pmatrix}
\]

   a) How many spanning trees has in the graph G? (3%)
   b) Draw one of its spanning trees. (1%)

11. Let H be the graph shown below where the edges represent the streets in a city. A police officer wants to make a trip to patrol each street exactly once. Can such a trip be designed and give your reason? (The police officer may start at any vertex in the graph). (5%)

\[\text{Graph Diagram}\]