(1) (a) Let $K$ be a compact set in $\mathbb{R}^3$ and define
\[ C = \{(x, y) \in \mathbb{R}^2 \mid \exists z \in \mathbb{R} \text{ such that } (x, y, z) \in K\}. \]
Prove that $C$ is a compact set in $\mathbb{R}^2$. (6 pts)

(b) Let $A$ be a path connected subset of $\mathbb{R}^n$ and $f : A \to \mathbb{R}^m$ be a continuous function. Define the graph of $f$ by
\[ G_f = \{(u, v) \in \mathbb{R}^{n+m} \mid u \in A, v = f(u)\}. \]
Prove that $G_f$ is path connected. (6 pts)

(2) (a) Find the following limit if it exists or show that it does not exist.
\[ \lim_{n \to \infty} \sum_{k=1}^{2n} \frac{1}{\sqrt{n^2 + 2kn}} = \lim_{n \to \infty} \left( \frac{1}{\sqrt{n^2 + 2n}} + \ldots + \frac{1}{\sqrt{n^2 + 4n^2}} \right). \]
Justify your assertion. (7 pts)

(b) Let $F : \mathbb{R} \to \mathbb{R}$ be the function defined by
\[ F(x) = \int_{x^3}^{0} \cos t - \frac{1}{t^2} \, dt. \]
Show that $F$ is differentiable and find $F'(x)$. (10 pts)

(3) (a) Prove that
\[ e^x \geq 1 + x \]
for all $x \geq 0$. (5 pts)

(b) Let $f_n : (0, \infty) \to \mathbb{R}$ and $g_n : (0, \infty) \to \mathbb{R}$ be two sequences of functions defined by
\[ f_n(x) = \sum_{k=1}^{n} \frac{e^{-kx}}{k\sqrt{k}}, \]
\[ g_n(x) = \sum_{k=1}^{n} \frac{e^{-kx}}{\sqrt{k}}. \]

(i) Prove that $f_n$ converges uniformly to a function $f$ on $(0, \infty)$. (5 pts)

(ii) Use Part (a) to prove that $g_n$ converges pointwise to a function $g : (0, \infty) \to \mathbb{R}$. (6 pts)

(iii) Prove that
\[ f'(x) = -g(x). \] (5 pts)
(4) Suppose \( f, g, h \) are three functions
\[
f, g, h : \mathbb{R}^3 \rightarrow \mathbb{R}
\]
and their first partial derivatives are continuous. Let \( c_g, c_h \) be two constants and
\[
S_g = \{(x, y, z) \in \mathbb{R}^3 \mid g(x, y, z) = c_g\},
\]
\[
S_h = \{(x, y, z) \in \mathbb{R}^3 \mid h(x, y, z) = c_h\}
\]
are level surfaces of \( g \) and \( h \) respectively. Moreover,
\[
\nabla g \neq \mathbf{0} \text{ on } S_g, \quad \nabla h \neq \mathbf{0} \text{ on } S_h.
\]
(a) Consider the problem of finding the extreme values of \( f(x, y, z) \)
subject to the constraint
\[
(x, y, z) \in S_g.
\]
If \( P = (x_0, y_0, z_0) \) is an extreme point, show that there exists a constant \( \lambda_g \) such that
\[
\nabla f(x_0, y_0, z_0) = \lambda_g \nabla g(x_0, y_0, z_0).
\]
(10 pts)
(b) Consider the problem of finding the extreme values of \( f(x, y, z) \)
subject to the constraint
\[
(x, y, z) \in S_g \cap S_h.
\]
If \( P = (x_0, y_0, z_0) \) is an extreme point, show that there exist constants \( \lambda_g \) and \( \lambda_h \) such that
\[
\nabla f(x_0, y_0, z_0) = \lambda_g \nabla g(x_0, y_0, z_0) + \lambda_h \nabla h(x_0, y_0, z_0).
\]
(10 pts)
(c) Find the maximum value of the function
\[
f(x, y, z) = x + 2y + 3z
\]
on the curve of intersection of the plane \( x - y + z = 1 \) and the cylinder \( x^2 + y^2 = 1 \). (10 pts)
(5) (a) State in detail the theorem of change of variables for a double integral. (10 pts)
(b) Evaluate the integral
\[
\iint_S e^{\frac{x+y}{x-y}} \, dA,
\]
where \( S \) is the trapezoidal region with vertices \((1, 0), (2, 0), (0, -2)\) and \((0, -1)\). (10 pts)