1. In the hospital, the calories and protein in the meals for the patients need to be carefully controlled. The following table shows the amount of calories and protein of eggs, milk, and juice, which will be used to design the breakfast.

<table>
<thead>
<tr>
<th></th>
<th>Calories</th>
<th>Protein (g)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 egg</td>
<td>75</td>
<td>6</td>
</tr>
<tr>
<td>1 cup milk</td>
<td>150</td>
<td>9</td>
</tr>
<tr>
<td>1 cup juice</td>
<td>100</td>
<td>2</td>
</tr>
</tbody>
</table>

a. Given that exact 525 calories and 24 grams of protein are required, formulate a system of linear equations to model this problem. (2%) 

b. Solve the system of linear equations derived in a. (4%) 

c. If a patient cannot eat milk due to allergy, design a special breakfast for him. (2%) 

2. Let \( T(\vec{x}) = A\vec{x} \) be a linear transformation from \( \mathbb{R}^3 \) to \( \mathbb{R}^3 \), where \( A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 2 & 5 & 2 \end{bmatrix} \).

a. If \( \vec{x}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \), find \( T(\vec{x}_1) \). (2%) 

b. Find the inverse of \( A \). (4%) 

c. If \( T(\vec{x}_2) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \), find \( \vec{x}_2 \) based on the inverse matrix derived in b. (2%) 

3. Assume the coordinate of any vector \( \vec{x} \) with respect to a basis \( \mathcal{B} \) is denoted as \( [\vec{x}]_{\mathcal{B}} \). Consider the basis \( \mathcal{B}1 \) of \( \mathbb{R}^2 \) consisting of vectors, \( \vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \) and \( \vec{v}_2 = \begin{bmatrix} 5 \\ 3 \end{bmatrix} \).

a. If \( [\vec{x}_1]_{\mathcal{B}1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \), find \( \vec{x}_1 \). (3%) 

b. If \( \vec{x}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \), find \( [\vec{x}_2]_{\mathcal{B}1} \). (3%) 

c. Consider another basis \( \mathcal{B}2 \) of \( \mathbb{R}^2 \) consisting of vectors, \( \vec{u}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \) and \( \vec{u}_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \). If \( [\vec{x}_3]_{\mathcal{B}2} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \), find \( [\vec{x}_3]_{\mathcal{B}1} \). (4%)
4. A subspace $E$ of $\mathbb{R}^2$ is spanned by two vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. If $\bar{x} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$, find the orthogonal projection of $\bar{x}$ onto $E$. (10%)

5. There are two major transportation modes between Taipei and Kaohsiung: air and ground (including private vehicles, inter-city buses, and trains). Assume the total number of passengers (air plus ground) is relatively stable, but a certain percentage of passengers change their transportation modes from year to year. This mode changing behavior can be represented by the following transition matrix:

<table>
<thead>
<tr>
<th></th>
<th>Ground</th>
<th>Air</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ground</td>
<td>0.8</td>
<td>0.2</td>
</tr>
<tr>
<td>Air</td>
<td>0.4</td>
<td>0.6</td>
</tr>
</tbody>
</table>

For example, for the next year, 60% of the current air passengers will stick to air transportation, but 40% of them will shift to ground transportation.

a. If the numbers of annual air and ground passengers for this year are 2 and 8 millions respectively, find the number of passengers for both modes in the next year. (2%)

b. Find the eigenvalues and eigenvectors of the transition matrix above. (8%)

c. Based on the answers in b., estimate the share of these two modes in the long run, if above transition matrix remains unchanged. (4%)

6. 求 $y = \ln \sqrt{u^2 + v^2}$ 求 $\mathbf{d}^2 y$ (以 $u \cdot v$ 的微分表示) (5%)

7. 求 $\int \frac{dx}{\sqrt{(x+1)^2(x-1)^4}}$ (5%)

8. 求 $\int_0^1 \frac{r^3}{\sqrt{4 + r^2}} \, dr$ (5%)

9. 設 $x^2 + 2y^2 + 3z^2 + xy - z - 9 = 0$ 求 $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y^2}$ 當 $x = 1$, $y = -2$, $z = 1$ 時的值. (10%)

10. 求由 $xy = a^2$, $x + y = \frac{5a}{2}$, (a>0)所圍之面積 (10%)

11. 請計算 $I = \oint (x + y)^2 \, dx - (x^2 + y^2) \, dy$, 其中 $k$ 經過 $A(1,1)$, $B(3,2)$, $C(2,5)$ 爲頂點的三角形周邊 (15%)