Problem 1 (15 points)

Consider the differential equation $y' = 2xy$ for unknown function $y(x)$.
(a) (5 %) What is the general solution for $y(x)$?
(b) (5 %) The solution can also be obtained by expanding $y(x)$ as a power series

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$ 

What is the recursive relation for the coefficients $a_n$?
(c) (5 %) Use the recursive relation to show that the power series is the same the result in (a).

Problem 2 (15 points)

Find the general solution with undetermined constants for the following differential equations.
(a) (7 %) $y'' - 5y' + 4y = 0$
(b) (8 %) $y'' - \frac{2}{x^2} = 0$

Problem 3 (20 points)

Consider the diffusion equation

$$\frac{\partial \rho}{\partial t} = D \frac{\partial^2 \rho}{\partial x^2}$$

for unknown function $\rho(x,t)$. $x$ is the space and $t$ is the time coordinates. The space Fourier transform of $\rho(x,t)$ is

$$R(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \rho(x,t)e^{ikx}dx.$$ 

Assume the initial condition is $\rho(x,0) = A\delta(t)$, where $\delta(t)$ is the Dirac delta-function.
(a) (5 %) What is the differential equation for $R(k,t)$?
(b) (5 %) What is the initial condition $R(k,0)$?
(c) (5 %) Use the results in (a) and (b) to show that $R(k,t) = \frac{A}{\sqrt{2\pi}} e^{-Dk^2t}$.
(d) (5 %) Use the result in (c) to get $\rho(x,t)$.
Problem 4 (20 points)

Consider the following potential

$$\phi(r) = \frac{1}{r^{2n}} - \frac{1}{r^n} \quad \text{for } r > 0,$$

where $n$ is a positive integer. The function $f_T = \phi''(r)$ has a minimum at $r_a$, and the function $f_L(r) = \phi'(r)/r$ has a maximum at $r_b$, where the prime denotes the derivative with respect to $r$.

(a) (10%) Express $r_a$ and $r_b$ in terms of $n$.

(b) (10%) Show that as $n$ is very large, the absolute value of $f_T(r_a)$ increases with $n^2$ and the value of $f_L(r_b)$ increases with $n$.

Problem 5 (20 points)

Use residue theorem to evaluate the following integrations

(a) (8%) \[ \int_0^\infty \frac{x^2}{x^4 + 1} \, dx \]

(b) (12%) \[ \int_0^\infty \frac{(\ln x)^2}{x^2 + 1} \, dx \]

Problem 6 (10 points)

The interaction between two equal-mass harmonic oscillators with frequency $\omega_1$ and $\omega_2$ is bilinear in their displacements, and their total potential has the following form

$$U(x_1, x_2) = \frac{1}{2} M \omega_1^2 x_1^2 + \frac{1}{2} M \omega_2^2 x_2^2 - I x_1 x_2,$$

where $M$ is the mass and $x_1$ and $x_2$ are their displacements.

(a) (6%) What are the normal-mode frequencies $\tilde{\omega}_1$ and $\tilde{\omega}_2$ of the system?

(b) (4%) Under what condition, the square of one of the normal-mode frequency ($\tilde{\omega}_1^2$ or $\tilde{\omega}_2^2$) will be negative?