1. Suppose the moment generating function for a random variable $X$ is given by

$$M_X(t) = \exp[-2 + 2 \cdot \exp(t)] \cdot [0.75 + 0.25 \cdot \exp(t)]^3.$$ 

(a) What is the distribution of $X$? (5%) 
(b) Find the probability $P(X > 1)$. (5%) 

2. If $X$ has a negative binomial distribution with probability mass function

$$P(x) = \binom{x-1}{r-1} p^r (1 - p)^{x-r}, x = r, r + 1, \ldots.$$ 

Find (a) $E[X]$. (10%)  
(b) $V[X]$. (10%)  
(c) the formulas for the method of moments estimates of $p$ and $r$. (10%) 

3. A random sample of size $n$ is taken from the probability density function 

$$f(y; \theta) = 2y\theta^2, 0 \leq y \leq 1/\theta.$$ 

Find the maximum likelihood estimate (MLE) of $\theta$. (10%) 

4. Let $X_1, \ldots, X_n$ be a random sample from the distribution that has the probability density function 

$$f(x; \theta) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{(x - \theta)^2}{2}\right\}, -\infty < x < \infty.$$ 

It is desired to test the null hypothesis $H_0: \theta = 0$ versus the alternative hypothesis $H_1: \theta = 1$.  
(a) Find a best critical region of size $\alpha$. (10%)  
(b) State the theorem you have used in (a). (6%) 

5. Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ be two independent samples from $N(\mu_1, \sigma^2)$ and $N(\mu_2, \sigma^2)$, respectively. Here $N(\mu, \sigma^2)$ denotes the normal distribution with mean $\mu$ and variance $\sigma^2$.  
(a) If $\sigma^2$ is assumed known, find a 100$(1 - \alpha)$% confidence interval for $\theta = \mu_1 - \mu_2$. (8%)  
(b) If $\sigma^2$ is unknown, construct the likelihood ratio test for testing the null hypothesis $H_0: \mu_1 = \mu_2$ versus $H_1: \mu_1 \neq \mu_2$. (10%)
6. Let $Y_i = \alpha + \beta X_i + \varepsilon_i$, $i = 1, \ldots, n$. If $(\varepsilon_1, \ldots, \varepsilon_n)$ follows a multivariate normal distribution with mean vector $0$ and covariance matrix $\sigma^2 I_n$, where $I_n$ is the identity matrix.

(a) What are the estimates of $\alpha$ and $\beta$? (10%)
(b) Are the estimates of $\alpha$ and $\beta$ obtained in (a) unbiased? (6%)