1. For a first-order linear differential equation
   \[ \frac{dy}{dx} + P(x)y = Q(x) \]  
   (10%)

   (a) Find a general form for the solution of y.
   (b) If \( P(x) = -2x, Q(x) = x \), find a complete solution of y.

2. (a) For a second-order differential equation
   \[ y'' - 3y' + 2y = x^2(e^x + e^{-x}) \]
   Write down the correct form of its particular solution. Do not solve.

   (b) Do the same as in part (a) for the following equation
   \[ y'' + 4y = x^2 \cos 2x \]

   (c) Find the particular solution \( y_p \) for the equation
   \[ x^2 y'' - xy' + y = x^{1/2} \]  
   (14%)

3. Determine the periodic solutions, if any, of the system
   \[ \frac{dx}{dt} = y + \frac{x}{\sqrt{x^2 + y^2}} (x^2 + y^2 - 2) \]
   \[ \frac{dy}{dt} = -x + \frac{y}{\sqrt{x^2 + y^2}} (x^2 + y^2 - 2) \]  
   (8%)

4. Consider the initial value problem
   \[ \frac{d^2 y}{dt^2} + y = g(t), \quad y(0) = 0, \quad y'(0) = 0 \]
   (10%)

   where
   \[ g(t) = u_0(t) + \sum_{k=1}^{n} (-1)^k u_{k\pi}(t) \]
   \[ u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases} \quad c \geq 0 \]

   (a) Find the solution of the initial value problem.
   (b) Investigate how the solution changes as \( n \) increases, What happens as \( n \to \infty \)?
5. Consider a cylinder of radius $a$ and height $H$. The base of the cylinder is at $z=0$ and the top is at $z=H$. Find a function $U(r,z,t)$ which satisfies:
\[
\frac{\partial U}{\partial t} = k\nabla^2 U
\]
in the domain and the stated boundary condition and initial condition. The boundary condition is that $U=0$ on the surface of the cylinder for all time. The initial condition is that $U$ within the domain $= \alpha(r)\beta(z)$ at time $t=0$. 

Hint: As it is assumed that $U$ does not depend on $\theta$, the $\theta$ term in the following cylindrical coordinate can be ignored.

\[
\nabla^2 f = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}
\]

6. (a) Give the definitions of Hermitian matrix and Normal matrix.
(b) Which of the following matrices are Hermitian? Normal?

\[
A = \begin{bmatrix} 1 & 2-i \\ 2+i & -1 \end{bmatrix} \quad B = \begin{bmatrix} \frac{1}{\sqrt{2}} & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \\
C = \begin{bmatrix} 0 & i & 1 \\ i & 0 & -2+i \\ -1 & 2+i & 0 \end{bmatrix} \quad D = \begin{bmatrix} 3 & 1+i & i \\ 1-i & 1 & 3 \\ -i & 3 & 1 \end{bmatrix}
\]

(12%) 

7. Find (a) the eigenvalues and eigenvectors of the matrix given by
\[
\begin{bmatrix} 1 & 0 & 0 \\ -8 & 4 & -6 \\ 8 & 1 & 9 \end{bmatrix}
\]

and (b) the coordinate vector of $[1,2,-2]$ relative to the ordered basis $([1,1,1], [1,2,0], [1,0,1])$ in space $\mathbb{R}^3$. 

(10%)
8. Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that
   
   $T(1, 0, 0) = (2, 3, 2)$
   $T(0, 1, 0) = (3, 3, 4)$
   $T(0, 0, 1) = (1, 1, 1)$.

   The inverse of $T$ is denoted by $T^{-1}$.
   
   (a) Find $T(1, 2, 3)$.
   (b) Find $T^{-1}(0, 1, 0)$.
   (c) Find $T^{-1}(1, 2, 3)$.

9. There exist two values of $\lambda$ delivering the nontrivial solutions in the linear system:

   $(\lambda - 1)x - 4y = 0$
   $-2x + (\lambda - 3)y = 0$

   Find the corresponding nontrivial solutions.

10. Find the projection of the vector $v = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ onto the subspace $S = \text{Span}\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \}$.

    (8%)