1. 下列 10 小題，自第(a) 小題至第(j) 小題為是非題。若該小題敘述正確，
請答“是”；若否，請答“非”；勿須對答案作任何解釋。每小題答對給
三分。答錯不但不給分，且答錯一題要倒扣本科總分十分。若不作答則
不給分但也不倒扣。

(a) Let \( x_1(t), 0 \leq t < \infty \), be a solution of the following differential equation

\[
\frac{d^2 x(t)}{dt^2} + \cos t \frac{dx(t)}{dt} + e^{-t} x(t) = e^{-t}
\]

on \([0, \infty)\), and let \( x_2(t), 0 \leq t < \infty \), be a solution of the associated homogeneous

equation on \([0, \infty)\), then \( c_1 x_1(t) + c_2 x_2(t) \) is also a solution of (1) on \([0, \infty)\) for

any real constants \( c_1 \) and \( c_2 \).

(b) Assume that a system is modeled by the following differential equation

\[
\frac{d^2 x(t)}{dt^2} + e^{-t} \frac{dx(t)}{dt} + e^{-2t} x(t) = e^{-3t}, \text{ where } x(t) \text{ and } \frac{dx(t)}{dt}
\]

denote the states of

the system at time \( t \). Suppose the system is initially at rest, i.e., \( x(0) = 0 \) and

\[
\frac{dx(t)}{dt} \bigg|_{t=0} = 0, \text{ then the system's state trajectory } (x(t), \frac{dx(t)}{dt}), 0 \leq t < \infty, \text{ is unique.}
\]

(c) Let \( x_1(t) \) be a non-zero solution of \( \frac{d^2 x(t)}{dt^2} + 5 \frac{dx(t)}{dt} + p(t)x(t) = 0 \) on an open

interval \( I \), where \( p(t) \) is a continuous function on \( I \). Suppose

\[ x_1(t) \neq 0, \forall t \in I. \] \( x_2(t) = x_1(t) \int \frac{e^{-3t}}{x_1(t)} dt \), then \( x_1(t) \) and \( x_2(t) \) are linearly

independent on \( I \).

(d) Let \( x_1(t), 0 \leq t < \infty \), be a solution of the following differential equation

\[
\sqrt{2} \frac{d^2 x(t)}{dt^2} + \sqrt{17} \frac{dx(t)}{dt} + \sqrt{8} x(t) = 0 \text{ with initial values } x_1(0) = 8, \frac{dx_1(t)}{dt} \bigg|_{t=0} = -4.
\]

There must exist a \( t > 0 \) such that \( x_1(t) < 0 \).
(e) \( t = 0 \) is an irregular singular point of the following differential equation:
\[
t^2(1+t)\frac{d^2x(t)}{dt^2} + (2 - t^2) \frac{dx(t)}{dt} + (2 + 3t)x(t) = 0.
\]

(f) If \( F(s) = \mathcal{L}\{f(t)\} \), the Laplace transform of \( f(t) \), exists for \( s > c \), then
\[\mathcal{L}\{e^{at}f(t)\} = F(s-a) \] also exists for \( s > c \), where both \( c \) and \( a \) are real numbers.

(g) If \( f(t) \) is a continuous function, then
\[\lim_{s \to \infty} \int_0^\infty e^{-st} f(t)dt = 0.\]

(h) \( \Phi(t) = \begin{bmatrix} e^{-2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{bmatrix} \) is the unique fundamental matrix of the following differential equations:
\[
\frac{dx_1(t)}{dt} = 4x_1(t) + 2x_2(t), \quad \frac{dx_2(t)}{dt} = 3x_1(t) - x_2(t).
\]

(i) Assume \( \hat{y}_1(t) \) and \( \hat{y}_2(t) \) satisfy \( \hat{y}_1(t)x_1(t) + \hat{y}_2(t)x_2(t) = 0 \) and
\[
\hat{y}_1(t)\frac{dx_1(t)}{dt} + \hat{y}_2(t)\frac{dx_2(t)}{dt} = \cos t, \text{ where } x_1(t) \text{ and } x_2(t) \text{ are two linearly independent solutions of}
\]
\[
\frac{d^2x(t)}{dt^2} + e^{-t} \frac{dx(t)}{dt} + e^{-2t}x(t) = 0. \text{ If}
\]
\[
y_1(t) = \int_0^t \hat{y}_1(\tau)d\tau, \quad i = 1, 2, \text{ then } y_1(t)x_1(t) + y_2(t)x_2(t) \text{ is a solution of}
\]
\[
\frac{d^2x(t)}{dt^2} + e^{-t} \frac{dx(t)}{dt} + e^{-2t}x(t) = \cos t \quad \text{such that } y_1(0)x_1(0) + y_2(0)x_2(0) = 0 \quad \text{and}
\]
\[
\left. \frac{d}{dt}[y_1(t)x_1(t) + y_2(t)x_2(t)] \right|_{t=0} = 1.
\]

(j) \[
\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} t^2e^{-t} + te^{-t} \\ -t^2e^{-t} - 5te^{-t} + e^{-t} \\ -t^2e^{-t} + te^{-t} + e^{-t} \end{bmatrix}
\]
is a solution of the first order linear differential equations
\[
\begin{bmatrix} \frac{dx_1(t)}{dt} \\ \frac{dx_2(t)}{dt} \\ \frac{dx_3(t)}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -7 & -3 & -5 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}.
\]
2. (10%) Let $A \in \mathbb{R}^{n \times n}$. It is known that one of the equivalent conditions for "$A$ is a nonsingular matrix" is "$\det(A) \neq 0$". Please give another five equivalent conditions.

3. (8%) Find the best least squares fit by a linear function, $y = ax + b$, to the data $(-1,2), (1,3), (2,5)$ and $(3,5)$.

4. (5%) Is the set $\{(x, \cos x) | x \in \mathbb{R}\}$ a subspace of $\mathbb{R}^2$? (MUST WITH REASON!)

5. Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 1 \\ 0 & 1 & 5 \end{pmatrix}$.
   
   (a) (8%) Find the eigenvalues and the associated eigenspaces of $A$.
   
   (b) (4%) Find the eigenvalues and the associated eigenspaces of the matrix $5A^{23} + 2A^{13} - 7A^{-1}$.

6. A function with complex variable is given as $f(z) = |z|^2 = zz^\ast$
   
   (c) (5%) Is the function continuous at $z = 0$? (Prove or disprove it)
   
   (d) (5%) Is the function differentiable at $z = 0$? (Prove or disprove it)
   
   (e) (5%) Is the function analytic at $z = 0$? (Prove or disprove it)

7. In Laplace transform theory, we have the transformation pair:
   
   (i) $L\{e^{-t} \sin(t)\} = \frac{1}{s^2 + 2s + 2}$ and
   
   (ii) $L^{-1}\left\{\frac{1}{s^2 + 2s + 2}\right\} = e^{-t} \sin(t)$

   (f) (10%) Derive (i) by performing the integration step-by-step.

   (g) (10%) Derive (ii) by performing complex integration.