Linear Algebra

1. For each statement below, indicate whether it is true or not. (5%)
   a) If $A$ is a symmetric matrix then $\text{rank}(A) = \text{rank}(A^T)$
   b) If $Ax$ and $Ay$ are linearly independent then $x$ and $y$ are linearly independent. ($A$ is a matrix and $x$ and $y$ are vectors.)
   c) If $AB = 0$ then the null space of $A$ contains the column space of $B$.
   d) If $A$ and $B$ are $2 \times 2$ matrices such that $ABAB = 0$ then $BABA = 0$
   e) If $A = BC$ where $B$ is a $5 \times 4$ matrix and $C$ is a $4 \times 5$ matrix then $A$ is invertible.

2. Let $F$ be the space of all functions from $\mathbb{R}$ to $\mathbb{R}$, $E$ be the subspace of $F$ that contains all functions satisfying $f(x) = f(-x)$, and $O$ be the subspace of $F$ that contains all functions satisfying $f(x) = -f(-x)$, show that $F = E + O$. (5%)

For problems 3~5, give your answers directly.
No explanation or computation is necessary.

3. Let $A = \begin{bmatrix} 0 & b & b+c \\ a & a+b & b \\ a & a & 0 \end{bmatrix}$, where $a \neq 0, b \neq 0, c \neq 0$.

Factor the matrix $A$ into $PA = LDL^T$, where $P$ is a permutation matrix, $L$ is a lower triangular matrix, and $D$ is a diagonal matrix. (5%)

4. Suppose the complete solution to the equation $Ax = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$ is $x = s \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Find $A$. (5%)

5. Define the square matrices
   $$A_n = \begin{bmatrix} a_1 & 1 & 0 & 0 \\ -1 & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & -1 \end{bmatrix}, \quad n \geq 1$$

and let $d_n = \det A_n$. Find a recurrence relation for $d_n$. (Don’t solve it.) (5%)
以下各計算題，只需寫出答案。

6. (1) In \( \mathbb{R}^2 \), let \( T \) be the orthogonal projection onto the subspace \( S \), where

\[
S = \text{span} \{ (1, 1, 0), (1, 1, 1) \}.
\]

Find the standard matrix representation \( A \) for this linear transformation.

(2) As in part (1), find the orthogonal projection of \((1, 0, 0)\) onto the subspace \( S \). (5%)

7. Let \( A = \begin{bmatrix} 0 & -2 & 3 \\ 4 & 0 & 11 \end{bmatrix} \) and let \( T \) be the linear transformation given by

\[ T(x) = Ax. \]

Find (1) kernel \((T)\), (2) nullity \((T)\), (3) range \((T)\), (4) rank \((T)\). (4%)

8. In \( \mathbb{R}^3 \), let \( B_1 \) and \( B_2 \) be two basis, where

\[
B_1 = \{ (a_1, b_1, c_1), (a_2, b_2, c_2) \} \quad \text{and} \quad B_2 = \{ (x_1, y_1, z_1), (x_2, y_2, z_2) \}
\]

Find the transition matrix from \( B_1 \) to \( B_2 \). (2%)

9. In \( \mathbb{R} \), find the rotation matrix that rotates a point 30° about the y-axis. (2%)

10. Are the following two matrices similar? If so, find a matrix \( P \) such that \( B = P^TP \).

\[
\begin{align*}
(1) \quad A &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} & B &= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
(2) \quad A &= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} & B &= \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}
\end{align*}
\]

(4%)

11. Find a basis \( B \) for the domain of \( T \) such that the matrix of \( T \) related to \( B \) is diagonal.

\[
T : \mathbb{R}^2 \to \mathbb{R} \quad T(x, y) = (x+y, -x+y).
\]

(4%)

12. (1) For an invertible matrix \( A \), prove that \( A \) and \( A^{-1} \) have the same eigenvectors.

How are the eigenvalues of \( A^{-1} \) related to the eigenvalues of \( A \)?

(2) Let \( A \) be a diagonalizable matrix with \( n \) real eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_n \).

What is the determinant of \( A \)? Prove your answer briefly. (4%)
1. Use strong induction to prove that every integer $n > 1$ is the product of finitely many primes. (7%)

2. A computer network consists of eight computers. Each computer is directly connected to at least one of the other computers. Use pigeonhole principle to show that there are at least two computers in the network that are directly connected to the same number of other computers. (5%)

3. Prove that big-$O$ estimates for the function $(x + 1)^n$ is $O(x^n)$. (5%)

4. The Fibonacci numbers, $f_0$, $f_1$, $f_2$, ..., are defined by equations $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n = 2, 3, 4, ...$.
   a) Give a recursive algorithm using pseudocode for computing Fibonacci number. (2%)
   b) Give an iterative algorithm using pseudocode for computing Fibonacci number. (2%)
   c) How many additions have been used to find $f_n$ with the recursive approach? (2%)
   d) How many additions have been used to find $f_n$ with the iterative approach? (2%)

For problems 5~12, give your answers directly.

No explanation or computation is necessary.

5. How many nonisomorphic partially ordered sets with three elements are there? (3%)

6. Consider the poset $(P(S), \subseteq)$, where $P(S)$ is the set of all subsets of the set $S = \{1, 2, 3, 4\}$. At least how many disjoint chains can the set $P(S)$ be expressed as their union? (3%)

7. For any positive integer $n$, let $\equiv_n$ be a binary relation on the set of integers such that $a \equiv_n b$ if $a \equiv b \pmod{n}$. Clearly, $\equiv_n$ is an equivalence relation. Given positive integers $m$ and $n$, find $k$ that satisfies $\equiv_n \cap \equiv_m \equiv_k$. (3%)

8. The inversion table $b_1, b_2, K, b_n$ of a permutation on $1, 2, K, n$ is defined by letting $b_k$ be the number of elements to the left of $k$ that are greater than $k$.

Let  

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2 1 0 2 1 0
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be the inversion table of a given permutation. What would be the inversion table of the permutation obtained by applying one bubble-sort pass to the given permutation? (3%)

(A bubble-sort pass interchanges $a_i \leftrightarrow a_{i+1}$ iff $a_i > a_{i+1}$ for $i = 1, 2, K, n-1$)
9. How many permutations on 1, 2, K, n (n ≥ 2) are sorted by at most two bubble-sort passes? (3%)

10. Let $P_c(x)$ be the number of ways of properly coloring (i.e. no two adjacent vertices have the same color) the vertices of the graph $G$ with $x$ or fewer colors. $P_c(x)$ is called the chromatic polynomial of $G$ (in terms of $x$).

Determine the chromatic polynomial of a tree with $n$ vertices. (3%)

11. Which is a legal chromatic polynomial? (3%)
   a) $x^4 - 4x^3 + 6x^2 - 4x + 1$
   b) $x^4 - 4x^3 + 6x^2 - 4x$
   c) $x^4 - 4x^3 - 3x^2 + 6x$
   d) $x^4 - 4x^3 + 6x^2 - 3x$
   e) $2x^4 - 4x^3 + 5x^2 - 3x$

12. A sequence $a_1, a_2, K, a_n, K$ is linear recursive of degree $k$ if there are constants $c_1, c_2, K, c_k$ such that $a_n = c_1a_{n-1} + c_2a_{n-2} + K + c_ka_{n-k}$ for $n > k$. Find the recursion that generates the linear recursive sequence: 1, 4, 9, 16, 25, K, $n^2$, K (4%)