1. Let $m$ be a positive integer and define a relation $R$ on the set $X$ of all nonnegative integers by $aRb$ if and only if $a$ and $b$ have the same remainder when divided by $m$.

(a) (5%) Prove that $R$ is an equivalence relation on $X$.
(b) (5%) How many different equivalence classes does $R$ give? (Explain your answer.)

2. A permutation on the set $S = \{1, 2, \ldots, n\}$ is a bijective function $f: S \to S$.

(a) (2%) How many permutations are there on $S$?
(b) (3%) How many permutations $f$ on $S$ satisfying $f(1) = 1$?
(c) (5%) How many permutations $f$ on $S$ satisfying $f(i) = i \text{ for all } i (1 \leq i \leq n)$? (Explain your answer.)

3. (a) (5%) Suppose $a_0 = 1$ and $a_n = \left(1 - \frac{1}{n+1}\right)a_{n-1}$ for $n \geq 1$. Evaluate $a_{2003}$.

(b) (5%) Suppose $b_1 = 0$, $b_2 = 1$, and $b_n = (n-1)(b_{n-1} + b_{n-2})$ for $n \geq 3$. Evaluate $b_n$.

(c) (5%) Evaluate $\lim_{n \to \infty} \frac{b_n}{n!}$ as a real number.

4. (a) (5%) Prove that if $n + 1$ numbers are chosen from the set $\{1, 2, \ldots, 2003n\}$, then there are always two chosen numbers such that the difference between them is at most 2002.
(b) (5%) A point $(x, y)$ in the plane is called an integer point if $x$ and $y$ are both integers.

Let $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$, $(x_4, y_4)$, and $(x_5, y_5)$ be five integer points.

Prove that the midpoint of the segment joining some pair of the five integer points is also an integer point.
(c) (5%) Find the number of nonnegative integers $n$ such that $2003 + n$ is a multiple of $n + 1$. 
5. Two colorings of the corners of a regular 5-gon are equivalent if they can become the same coloring by using rotations and reflections of the regular 5-gon. For example, the following two colorings are equivalent.

Two colorings are inequivalent if they are not equivalent.
(a) (4%) How many inequivalent colorings are there when the corners of a regular 5-gon is colored with two colors?
(b) (7%) How many inequivalent colorings are there when the corners of a regular 5-gon is colored with six colors? (Explain your answer.)

6. (a) (4%) Let $r_n$ be the number of a $2 \times n$ matrices

$$
\begin{bmatrix}
  c_{11} & c_{12} & \cdots & c_{1n} \\
  c_{21} & c_{22} & \cdots & c_{2n}
\end{bmatrix}
$$

that can be formed by the numbers $1, 2, 3, 4, \ldots, 2n$ so that

$c_{11} < c_{12} < \cdots < c_{1n},$

$c_{21} < c_{22} < \cdots < c_{2n},$

and $c_{11} < c_{21}, c_{12} < c_{22}, c_{13} < c_{23}, \ldots, c_{1n} < c_{2n}$. Find $r_2$.

(b) (7%) Let $s_n$ be the number of sequences $d_1, d_2, \ldots, d_{2n}$ that can be formed by

$1, 1, \ldots, 1$ (共 $n$ 個 $1$) and $-1, -1, \ldots, -1$ (共 $n$ 個 $-1$) so that $d_1 + d_2 + \ldots + d_k \geq 0$ for all $k = 1, 2, \ldots, 2n$. Prove that $s_n = r_n$. 
7. (a) (4%) Determine whether the following two graphs $H_1$ and $H_2$ are isomorphic. (Explain your answer.)

\[ \begin{align*}
H_1 & \\
H_2 &
\end{align*} \]

(b) (7%) Prove that a simple graph $G$ of order $n$ with at least \( \frac{n^2 - 1}{2} \) edges is connected.

8. (a) (6%) Let $G$ be a simple graph with a finite number of vertices. Prove that if the degree of each vertex in $G$ is at least 2, then $G$ contains a cycle.

(b) (4%) The eccentricity of a vertex $v$ in a tree $T$ is the maximum length of a path beginning at $v$. A vertex $v$ in a tree $T$ is a center for $T$ if the eccentricity of $v$ is minimum. Find the center(s) of the following tree.

\[ \begin{align*}
\text{center(s) } & \\
\end{align*} \]

(c) (7%) Prove that a tree has at most two centers.