Notations: Let $R$ be the field of real numbers; $R^n$ the vector space of dimension $n$ over $R$; $P_n(R)$ the family of all polynomials with real coefficients and with degree at most $n$, and $M_n(R)$ the family of all $n \times n$ matrices over $R$.

1. (10%) Are $U = \{(x, y, z) \in R^3 \mid x + 2y - 3z + 3x + 2y - z = 0\}$,

$$V = \{(x, y, z) \in R^3 \mid \begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 1 \end{bmatrix} = 0\},$$

$$W = \{(x, y, z) \in R^3 \mid \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0\}$$

subspaces of $R^3$, justify your answer. Find their dimensions over $R$, give their geometric interpretations if they are.

2. (10%) Show that the polynomials $f(x) = 1$, $g(x) = x - 1$, and $h(x) = (x - 1)^2$ form a base of $P_2(R)$ over $R$. Find the coordinate of the vector $\alpha(x) = 2x^2 - 5x + 6$ relative to this base.

3. (15%)

a. (5%) Find the dimension of $U \cap W$ over $R$, where

$$U = \{(x_1, x_2, x_3, x_4, x_5) \in R^5 \mid x_1 - x_3 - x_4 = 0, x_1 + x_2 - 2x_3 - x_4 = 0\},$$

and

$$W = \{(x_1, x_2, x_3, x_4, x_5) \in R^5 \mid x_2 = x_3 = x_4, x_1 + x_5 = 0\}.$$

b. (5%) If $T$ is a linear transformation from $R^2$ into $R^3$ such that $T(1, 3) = (2, 1, \alpha)$, and $T(2, 1) = (3, \beta, 6)$, determine $(\alpha, \beta)$ so that $T$ is not one to one.

c. (5%) If the eigenvalues of $A \in M_2(R)$ are 0 and 1, and their corresponding eigenvectors are $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ respectively, find the matrix $A$.

4. (15%) Let $T(f(x)) = f(2) - f(1)x + \frac{1}{2}f''(0)x^2$ be a function from $P_2(R)$ into itself. Find $T^{36}(x^2 + x + 1)$, $T^{49}(4x^2 + 19x + 3)$ respectively; and then find a general expression for $T^n(f(x))$. 
5. (12%) (2%) Given $A, B \in M_n(R)$, what does it mean to say that $A$ is similar to $B$?

Write $A \sim B$ if $A$ is similar to $B$.

b. (5%) Show that $\sim$ is an equivalence relation.

c. (5%) Suppose that $A \sim B$. Is it true that $\det A = \det B$? Is it true that $A^k \sim B^k$ for all $k = 1, 2, 3, \ldots$? Answer true or false. If true, prove it. If false, give a counterexample with $n = 2$.

6. (14%) Suppose $A \in M_n(R)$ is orthogonal, namely $A^T A = I$. Here $A^T$ is the transpose of $A$.

a. (3%) What are the possible values of $\det A$?

b. (5%) Suppose that $Av = \alpha v$ for some nonzero vector $v$. What are all the possible values of the scalar $\alpha$? Explain with proof and examples.

c. (6%) Decide if each of the matrices $A^{-1}, 2A, A^2$ is orthogonal, answer yes or no. If yes, prove it. If no, no proof is needed.

7. (10%) Let $A = (a_{ij}) \in M_n(R)$ with $\det A = 5$. Suppose that

$$b_{ij} = a_{2j}, \quad b_{ij} = a_{ij}, \quad b_{ij} = a_{ij} \quad \text{for all } i \geq 3, j \geq 1.$$  

$$c_{ij} = a_{ij}, \quad d_{ij} = 3a_{ij} \quad \text{for all } i, j \geq 1.$$  

$$e_{ii} = e_{i2} = a_{ii}, \quad e_{ij} = a_{ij} \quad \text{for all } i \geq 1, j \geq 3.$$  

$$f_{ij} = 2a_{ij} + 3a_{i2}, \quad f_{ij} = a_{ij} \quad \text{for all } i \geq 2, j \geq 1.$$  

Find the determinants of $B = (b_{ij}), C = (c_{ij}), D = (d_{ij}), E = (e_{ij}), F = (f_{ij})$. No proof is needed.

8. (14%) Let $V$ be a vector space over $R$ with basis $\{s, t, u, v, w\}$, and $T : V \to V$ a linear transformation. Suppose that exactly 3 vectors of $\{0, u, v, w, 2v + 3w\}$ are in the image of $T$.

a. (5%) What are the possible values of $\text{rank } T$? Explain.

b. (9%) Suppose in addition that $T^2 = 0$. What is the nullity of $T$? Give a nonzero vector in the kernel of $T$. Explain.