Consider the following problem:

Maximize $Z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$

Subject to

\begin{align*}
    a_{11} x_1 + \ldots + a_{1n} x_n &= b_1 \\
    a_{21} x_1 + \ldots + a_{2n} x_n &= b_2 \\
    &\vdots \\
    a_{m1} x_1 + \ldots + a_{mn} x_n &= b_m
\end{align*}

Under what conditions does this problem have a bounded optimal solution?

Consider the following problem.

Maximize $Z = c_1 x_1 + c_2 x_2 + c_3 x_3$

Subject to

\begin{align*}
    x_1 + 2x_2 + x_3 &\leq b \\
    2x_1 + x_2 + 3x_3 &\leq 2b \\
    \text{and } x_1 &\geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0
\end{align*}

Note that values have not been assigned to the coefficients in the objective function $(c_1, c_2, c_3)$, and that the only specification for the right-hand side of the functional constraints is that the second one $(2b)$ be twice as large as the first $(b)$.

Now suppose that your boss has inserted her best estimate of the values of $c_1$, $c_2$, $c_3$, and $b$ without informing you and then has run the simplex method. You are given the resulting final simplex tableau below (where $x_4$ and $x_5$ are the slack variables for the respective functional constraints), but you are unable to read the values of $e$, $f$, $g$, $h$, and $Z^*$.

<table>
<thead>
<tr>
<th>Basic Variables</th>
<th>Equation</th>
<th>Coefficient of:</th>
<th>Right side</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$X_1$ $X_2$ $X_3$ $X_4$ $X_5$</td>
<td>$Z^*$</td>
</tr>
<tr>
<td>$Z$ (0)</td>
<td></td>
<td>7/10 0 0 3/5 4/5</td>
<td>$Z^*$</td>
</tr>
<tr>
<td>$X_2$ (1)</td>
<td></td>
<td>1/5 1 0 e f</td>
<td>1</td>
</tr>
<tr>
<td>$X_3$ (2)</td>
<td></td>
<td>3/5 0 1 g h</td>
<td>3</td>
</tr>
</tbody>
</table>

(a) Show your method to identify the value of $(c_1, c_2, c_3)$ that was used. (11 %)
(b) Show your method to identify the value of $b$ that was used. (3%)
(c) Calculate the value of $Z^*$ in two ways, where one way uses your results from part (a) and the other way uses your result from part (b). Show your two methods for finding $Z^*$. (6%)
A3. (7%)

A shoe company forecasts the following demands during the next three months: month 1, 200; month 2, 310; month 3, 240. It costs $7 to produce a pair of shoes with regular-time labor (RT) and $11 with overtime labor (OT). During each month, regular production is limited to 200 pairs of shoes, and overtime production is limited to 100 pairs. It costs $2 per month to hold a pair of shoes in inventory. Formulate a balanced transportation problem to minimize the total cost of meeting the next three months of demand on time. (不必求解)。
B1. If every node as well as every arc has a length, can you find the shortest path between two nodes? Explain and give examples. (9%)

B2. Consider a project consisting of nine jobs (A, B, ..., I) with the precedence relations and time estimates shown in the following table.

<table>
<thead>
<tr>
<th>Job</th>
<th>Predecessors</th>
<th>Optimistic</th>
<th>Most likely</th>
<th>Pessimistic</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>-</td>
<td>2</td>
<td>5</td>
<td>8</td>
</tr>
<tr>
<td>B</td>
<td>A</td>
<td>6</td>
<td>9</td>
<td>12</td>
</tr>
<tr>
<td>C</td>
<td>A</td>
<td>6</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>D</td>
<td>B, C</td>
<td>1</td>
<td>4</td>
<td>7</td>
</tr>
<tr>
<td>E</td>
<td>A</td>
<td>8</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>F</td>
<td>D, E</td>
<td>5</td>
<td>14</td>
<td>17</td>
</tr>
<tr>
<td>G</td>
<td>C</td>
<td>3</td>
<td>12</td>
<td>21</td>
</tr>
<tr>
<td>H</td>
<td>F, G</td>
<td>3</td>
<td>6</td>
<td>9</td>
</tr>
<tr>
<td>I</td>
<td>H</td>
<td>5</td>
<td>8</td>
<td>11</td>
</tr>
</tbody>
</table>

a) Construct the project network. (6%)
b) Find the critical path of the network. (4%)
c) Find the mean and standard deviation of the project duration. (6%)

B3. Explain the following terms in detail.
   a) max-flow min-cut theorem (3%)
   b) KKT conditions (5%)
C1 20%

At time $t=0$ customer A places a request for service and finds all $m$ servers busy and $n$ other customers waiting for service in an $M/M/m$ queueing system. All customers wait as long as necessary for service, waiting customers are served in order of arrival, and no new requests for service are permitted after $t=0$. Service times are assumed to be mutually independent, identical, exponentially distributed random variables, each with mean duration $1/\mu$.

(a) Find the expected length of time customer A spends waiting for service in the queue. 4%

(b) Find the expected length of time from the arrival of customer A at $t=0$ until the system becomes completely empty (all customers completely service). 4%

(c) Let $X$ be the order of completion of service of customer A; that is $X=k$ if A is the $k$-th customer to complete service after $t=0$. Find $P[X=k] (k=1,2,\ldots,m+n+1)$. 4%

(d) Find the probability that customer A completes service before the customer immediately ahead of him in the queue. 4%

(e) Let $w'$ be the amount of time customer A waits for service. Find $P[w'>x]$. 4%

C2 13%

Consider the following "fixed-charge" problem.

Maximize $Z=3x_1+7x_2+6f(x_3)$,

Subject to

\[ x_1 + 3x_2 + 2x_3 \leq 6 \]

\[ x_1 + x_2 \leq 5 \]

and \[ x_1 \geq 0, \ x_2 \geq 0, \ x_3 \geq 0 \]

where \[ f(x_3) = \begin{cases} 
0 & \text{if } x_3 = 0 \\
-1+x_3 & \text{if } x_3 > 0.
\end{cases} \]

Use dynamic programming to solve this problem.