1. Let $L=\{w \mid w \in \{a,b\}^*, \#_a(w)=\#_b(w)+1\}$, where $\#_\sigma(w)$ is the number of occurrences of symbol σ in string w.
   (a) (10%) Design a context-free grammar G such that $L(G)=L$.
   (b) (3%) Use your grammar to derive string *aaabbaa* and construct its derivation tree.

2. A ***k-coloring*** of an undirected graph $G=(V,E)$ is a function $c: V \to \{0,1,\ldots,k\}$ such that $c(u) \neq c(v)$ for every edge $(u,v) \in E$. What is the minimum number of colors to color the following graphs.
   (a) (2%) A complete graph of n vertices.
   (b) (2%) An n-vertex graph without cycles of odd length.
   (c) (2%) A planar graph of n vertices.

3. An ***Euler tour*** of a connected directed graph $G=(V,E)$ is a cycle that traverses each edge of G exactly once, although it may visit a vertex more than once.
   (a) (5%) Show that G has an Euler tour if and only if *in-degree(v)=out-degree(v)* for each vertex $v \in V$.
   (b) (6%) Describe an $O(|E|)$-time algorithm to find an Euler tour for a given graph.

4. Given two sequences X and Y, we say that a sequence Z is a ***common sequence*** of X and Y if Z is a subsequence of both X and Y. For example, if $X=(A, B, C, B, D, A, B)$ and $Y=(B, D, C, A, B, A)$ then (B, C, A) is a common sequence of X and Y.
   (a) (3%) Find a **longest** common sequence of sequences $(1, 0, 0, 1, 0, 1, 0, 1)$ and $(1, 0, 1, 1, 1, 0, 0, 1)$.
   (b) (7%) Describe an $O(mn)$-time algorithm for computing a **longest** common sequence of two sequences of length m and n, respectively.

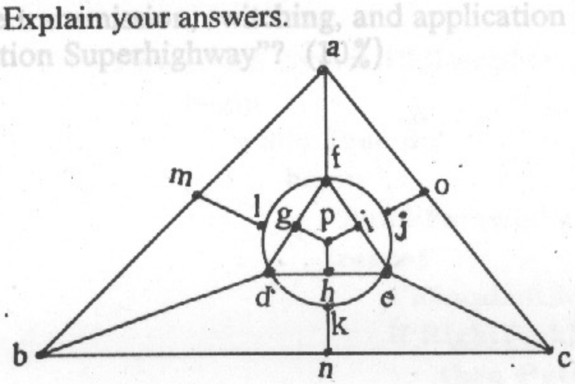
5. Let A be a nonempty set and let f be a function from A into A. We defined a binary relation R on A as: $R=\{(a, b) \mid f(a)=b\}$.
   (a) (5%) Find a function f so that R is an equivalence relation.
   (b) (5%) Show that the function f in (a) is unique.

6. The sequence $F_0, F_1, F_2, \ldots$, defined as: $F_0 = 0$, $F_1 = 1$, and $F_{i+2} = F_{i+1} + F_i$, $i \geq 0$.
   The sequence $N_0, N_1, N_2, \ldots$, defined as $N_0 = 0$, $N_1 = 1$, and $N_{i+2} = N_{i+1} + N_i + 1$, $i \geq 0$. Use mathematical induction to prove that $N_i = F_{i+2} - 1$ for all $i \geq 0$. (10%)

7. Let (k, *) be a subgroup of a finite group (G, *). Let S={ a*k | a ∈ G}.
   Definite (a * k) * (b* k) = (a *b) * k for a*k, b*k ∈ S.
   Let |x| denote the number of elements in x.
   (a) (3%) Find |S| in terms of |k| and |G|.
   (b) (7%) Show that (S, *) is a group.

8. Let A be a set with n distinct elements. How many different total ordering relations on A are there? (5%)

9. (a) (5%) Does the following graph have an Euler path?
   (b) (5%) Does the following graph have a Hamiltonian path?
   Explain your answers.



10. (7%)

$(x^2+x+1)(ax+b)^{-1} = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$

for $a \neq 0$ and $b \neq 0$. Find a_{10} and a_{20}.

11. We are given a red box, a blue box, and a green box. We are also given r red balls, b blue balls, and g green balls, balls of the same color are considered identical.
   Consider the following constraints:
   (1) No box contains a ball that has the same color as the box.
   (2) No box is empty.
   Determine the number of ways in which we can put the **r+b+g** balls into boxes so that
   (a) (2%) No constraints has to be satisfied.
   (b) (2%) Constraints 1 is satisfied.
   (c) (2%) Constraints 2 is satisfied.
   (d) (2%) Constraints 1 and 2 are satisfied.