機率

1. A box contains 3 red and 5 blue balls.
   (a) Balls are drawn at random without replacement until a red ball is drawn.
       What is the probability that exactly 3 drawn are required. (4 points)
   (b) Balls are drawn at random with replacement until a red ball is drawn. What is
       the probability that exactly 3 drawn are required. (4 points)
   (c) Balls are drawn at random with replacement 6 times. What is the probability
       that 3 red balls and 3 blue balls are drawn in these 6 times. (4 points)

2. Let $X$, $Y$ be independently uniformly distributed over $(0, 1)$.
   Define $Z = x^2$, and $W = \max(X,Y)$.
   (a) Find distribution function of $Z$. (3 points)
   (b) Find $E(Z)$. (4 points)
   (c) Find distribution function of $W$. (3 points)
   (d) Find $E(W)$. (4 points)

3. Let the joint density function of random variables $X$ and $Y$ be given by
   
   \[ f(x, y) = \begin{cases} 
   2 & \text{if } 0 \leq y \leq x \leq 1 \\
   0 & \text{otherwise} 
   \end{cases} \]
   (a) Calculate the marginal density function of $X$. (5 points)
   (b) Calculate $P(2X+2Y < 3)$. (7 points)

4. Let $X_1, X_2, \ldots, X_n$ be a random sample of size $n$ from a continuous distribution
   function $F$ with mean $\mu$ and variance $\sigma^2$. Let the sample mean
   \[ \overline{X} = \frac{X_1 + X_2 + \cdots + X_n}{n}. \]
   (a) What are $E(\overline{X})$ and $\text{Var}(\overline{X})$? (5 points)
   (b) Let $\mu = 100$, $\sigma^2 = 40$, $n = 90$. Use central limit theorem to approximate
       \[ P(99 < \overline{X} < 101). \] (Express your solution by using the standard normal
       distribution function $\Phi$.) (7 points)
線性代數

5. (3 points;複選，答對每個選項得1分，答錯每個選項扣1分，本題合計得分為負時，
   以0分計；未作答亦以0分計) Assume $A$ is an $m \times n$ matrix with rank $r$ and $b$
   is a column vector. Which statements are true?
   A. If $m > r$ and $n = r$, then $Ax = b$ must have no solution for some $b$ and
      exactly one solution for other $b$.
   B. If $m > r$ and $n > r$, then $Ax = b$ has infinitely many solutions for some $b$ and
      exactly one solution for other $b$.
   C. If $n = r$, then $Ax = b$ has either one solution or none.

6. (5 points;複選，答對每個選項得1分，答錯每個選項扣1分；本題合計得分為負時，
   本題以0分計；未作答亦以0分計) Suppose $Q = [q_1 \ q_2 \ q_3] = \begin{bmatrix} 1 & 1 & 1 \\
   1 & 1 & -1 \\
   1 & -1 & 1 \\
   1 & -1 & -1 \end{bmatrix}$. Let
   $S_{12} = \text{span}(q_1, q_2)$ and $S_{23} = \text{span}(q_2, q_3)$. Which statements are true?
   A. The union of the two subspaces $S_{12}$ and $S_{23}$ forms a vector space.
   B. The intersection of the two subspaces $S_{12}$ and $S_{23}$ forms a vector space.
   C. The span($q_1$) is an orthogonal complement of the subspace $S_{23}$.
   D. The rows of $Q$ form a basis for the row space.
   E. The dimension of the row space of $Q$ is 3.

7. (4 points;複選，答對每個選項得1分，答錯每個選項扣1分；本題合計得分為負時，
   以0分計；未作答亦以0分計) Which statements are correct?
   A. Assume $V$ and $W$ are vector spaces and $L : V \to W$ is a linear transformation.
      Let $\ker(L)$ denote the kernel of $L$ and $L(S)$ denote the image of $S$ for any
      subspace $S$ of $V$. If $\dim(V) = n$ and $\dim(W) = m$, then
      $\dim(\ker(L)) + \dim(L(V)) = m$. (Assume $n$ and $m$ are finite.)
   B. Using the same notations in the previous question, if $x \in \ker(L)$, then
      $L(v + x) = L(v)$ for any $v \in V$.
   C. Let $P_3$ be the space consisting of all polynomial of degree no more than 3, and
      $D$ be the differentiation operator on $P_3$. Then, $\ker(D) = \{0\}$.
   D. If $A$ and $B$ are similar matrices, then $\det(A - \lambda I) = \det(B - \lambda I)$ for any scalar
      $\lambda$. 
8. (6 points; 複選，答對每個選項得 2 分，答錯每個選項扣 2 分；本題合計得分為負時，本題以 0 分計；未作答亦以 0 分計) Which statements are correct?
   A. Let \( \mathbf{u}_1 = (-1, 2, 1), \ \mathbf{u}_2 = (1, 1, -2), \ \mathbf{v} = (10, 5, 10), \) and \( S = \text{span}(\mathbf{u}_1, \mathbf{u}_2). \) The (shortest) distance between \( \mathbf{v} \) and \( S \) is \( \frac{12\sqrt{50}}{5}. \)
   B. For the same setting in the previous question, the least square solution of the system \( x_1 \mathbf{u}_1 + x_2 \mathbf{u}_2 = \mathbf{v} \) is \( x_1 = \frac{1}{2} \) and \( x_2 = -\frac{1}{2}. \)
   C. Let \( V \) be an inner product space, and \( \langle \mathbf{u}_1, \mathbf{u}_2 \rangle \) denote the inner product of any two vectors \( \mathbf{u}_1, \mathbf{u}_2 \in V. \) If \( \mathbf{B} = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\} \) is an ordered basis of \( V, \) then for any vector \( \mathbf{u} \in V, \) the coordinate of \( \mathbf{u} \) can be given by \( [\mathbf{u}]_\mathbf{B} = \begin{bmatrix} \langle \mathbf{u}, \mathbf{v}_1 \rangle \\ \langle \mathbf{u}, \mathbf{v}_2 \rangle \\ \vdots \\ \langle \mathbf{u}, \mathbf{v}_n \rangle \end{bmatrix}. \)

9. (6 points; 複選，答對每個選項得 2 分，答錯每個選項扣 2 分；本題合計得分為負時，本題以 0 分計；未作答亦以 0 分計) Which statements are correct?
   A. Assume \( \mathbf{A} \) is a \( m \times n \) matrix and \( \mathbf{B} \) is a \( m \times p \) matrix. If \( \mathbf{X} \) is an \( n \times p \) unknown matrix, then the system \( \mathbf{A}^T \mathbf{AX} = \mathbf{A}^T \mathbf{B} \) always has a solution. (Here we assume \( m \geq n. \))
   B. The matrix \( \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \) possesses a complete orthonormal set of eigenvectors.
   C. \( \begin{bmatrix} 2 & 0 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \) is not defective.

10. (6 points; 複選，答對每個選項得 2 分，答錯每個選項扣 2 分；本題合計得分為負時，本題以 0 分計；未作答亦以 0 分計) Which statements are true?
   A. Assume \( \mathbf{A}_{3 \times 3} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \) and \( \mathbf{B}_{3 \times 3} = \begin{bmatrix} 2a_2 \end{bmatrix} a_2^T + a_3^T \). If \( \det \mathbf{A} = 2, \) then \( \det(\mathbf{AB}^{-1}) = 1. \)
   B. If \( \mathbf{P}_{3 \times 3} \) is a projection matrix that projects any vector in \( \mathbb{R}^3 \) onto the vector
u = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \text{ then there must be two eigenvectors that correspond to the eigenvalue of } 0.

C. If \( A \) is a \( 3 \times 3 \) matrix with 3 distinct eigenvalues 0, 1, 2, then the matrix \( (A + I) \) must be invertible.

11. (10 points) Suppose there is an election every year in a country and the total population of this country remains fixed. If 60% of the people voted for K Party whereas 40% of the people voted for D Party in the election last time. However, 8% of K Party voters and 4% of D Party voters change their minds and vote for the rival party each year. What will the percentages of K Party and D Party voters be after \( n \) years, when \( n \) approaches infinity?

12. Consider the vector space \( \mathbb{C}[0,1] \) with inner product defined by
\[
\langle f, g \rangle = \int_0^1 f(x)g(x)\, dx,
\]
where \( \mathbb{C}[0,1] \) denotes the set of all real-valued functions that are defined and continuous on the closed interval \([0,1]\).

A. (5 points) Use the Gram-Schmidt process to find an orthonormal basis for the subspace \( S \) spanned by 1, and \( x \).

B. (5 points) Find the best least squares approximation to \( e^x \) on the interval \([0,1]\) by a function in \( S \).