Discrete Mathematics There are two problems in the discrete mathematics part. The first problem consists of 6 independent questions, as 1.1, 1.2, 1.3, 1.4, 1.5 and 1.6. The second problem consists of 8 questions. Please answer them in order. For each question, you need to write down your answer first and then explain why.

Problem 1.

1.1. (5 points) When we toss a coin, we obtain either head or tail. Now we toss a coin 5 times. There are $2^5$ possible outcomes. How many of them contain no two consecutive heads?

1.2. (4 points) Let $A$ be the set $\{1, 2, 3\}$ and $B$ be the set $\{3.14, 2.71\}$. Let the notation $2^X$ denote the set of all subsets of $X$ (assume $X$ is a set). Let the notation $X \times Y$ denote the Cartesian product of the two sets $X$ and $Y$. How many elements are there in the set $2^{A \times B}$? You need to write down your answer first and explain why.

1.3. (4 points) How many partitions are there on a set of 4 elements?

1.4. (4 points) What is the smallest (positive) number $n$ satisfying:

- When divided by 2, the result is a square.
- When divided by 3, the result is a cube.

1.5. (4 points) Let $a, b, c, d$ be positive integers. Assume $a^3 = b^2$ and $c^3 = d^2$. If $c - a = 25$, what are $a, b, c, d$?

1.6. (4 points) Let $a, b$ be two symbols. The notation $a^3$ denotes the string $aaa$, that is, a string of three $a$'s. Similarly, the notation $a^4$ denotes the string of four $a$'s. Similarly, the notation $a^k$ denotes the string of $k$ $a$'s. Find a 1-1 mapping from $\mathcal{N}$ to $\{a^k b^j \mid j, k \in \mathcal{N}\}$. 
Problem 2.

2.1 (3 points) The complement of a simple graph G is the simple graph \( \overline{G} \) with the same vertices as G. An edge exists in \( \overline{G} \) if and only if it does not exist in G. If G is a simple graph with 11 edges and its complementary graph \( \overline{G} \) has 10 edges, then how many vertices does G have?

2.2 (3 points) Is there a unique binary tree with 6 vertices whose preorder vertex listing is ABCEFD and whose inorder vertex listing is ACFEBD. Justify your answer.

2.3 (3 points) \( N_h \) is defined as the minimum number of vertices in a balanced binary tree of height h. Find \( N_2 \), \( N_3 \).

2.4 (3 points) Let \( n(T) \) denote the number of vertices in a full binary tree T and \( h(T) \) the height of T. Find the value range of \( n(T) \) in terms of \( h(T) \).

2.5 (3 points) Under what conditions is an edge in a connected graph G contained in every spanning tree of G.

2.6 (3 points) Let \( a_r \) denote the number of bacteria there are on the \( r \)th day in a controlled environment. We define the rate of growth on the \( r \)th day to be \( a_r-2a_{r-1} \). If the rate of growth doubles every day, formulate the recurrence relation \( a_r \), given that \( a_0 = 1 \).

2.7 (3 points) \( F_n \) is the nth Fibonacci number, where \( n \) is a positive number. Compute \( F_{n+1}F_{n-1}-(F_n)^2 \)

2.8 (4 points) Let \((A, \star)\) and \((B, \bullet)\) be two algebraic systems with operators \( \star \) and \( \bullet \) defined on A and B, respectively. \( f \) is called a homomorphism from \((A, \star)\) to \((B, \bullet)\) if there exists a function \( f \) from A onto B such that for any \( x \) and \( y \) in A \( f(x \star y) = f(x) \bullet f(y) \). Justify briefly whether such a homomorphism exists.

\[
\begin{array}{cccc}
\star & a & b & c \\
\hline
a & a & a & d \\
b & b & a & c \\
c & c & b & a \\
d & d & d & b \\
\end{array}
\]

\[
\begin{array}{ccc}
\bullet & \alpha & \beta \\
\hline
\alpha & \alpha & \beta \\
\beta & \beta & \alpha \\
\end{array}
\]
3. (3 points; 複選，答對每個選項得1分，答錯每個選項扣1分；本題合計得分為負時，以0分計；未作答亦以0分計) Assume $A$ is an $m \times n$ matrix with rank $r$ and $b$ is a column vector. Which statements are true?

A. If $m > r$ and $n = r$, then $Ax = b$ must have no solution for some $b$ and exactly one solution for other $b$.

B. If $m > r$ and $n > r$, then $Ax = b$ has infinitely many solutions for some $b$ and exactly one solution for other $b$.

C. If $n = r$, then $Ax = b$ has either one solution or none.

4. (5 points; 複選，答對每個選項得1分，答錯每個選項扣1分；本題合計得分為負時，本題以0分計；未作答亦以0分計) Suppose $Q = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix}$. Let $S_{12} = \text{span}(q_1, q_2)$ and $S_{23} = \text{span}(q_2, q_3)$. Which statements are true?

A. The union of the two subspaces $S_{12}$ and $S_{23}$ forms a vector space.

B. The intersection of the two subspaces $S_{12}$ and $S_{23}$ forms a vector space.

C. The span$(q_1)$ is an orthogonal complement of the subspace $S_{23}$.

D. The rows of $Q$ form a basis for the row space.

E. The dimension of the row space of $Q$ is 3.

5. (4 points; 複選，答對每個選項得1分，答錯每個選項扣1分；本題合計得分為負時，以0分計；未作答亦以0分計) Which statements are correct?

A. Assume $V$ and $W$ are vector spaces and $L: V \to W$ is a linear transformation. Let $\ker(L)$ denote the kernel of $L$ and $L(S)$ denote the image of $S$ for any subspace $S$ of $V$. If $\dim(V) = n$ and $\dim(W) = m$, then $\dim(\ker(L)) + \dim(L(V)) = m$. (Assume $n$ and $m$ are finite.)

B. Using the same notations in the previous question, if $x \in \ker(L)$, then $L(v + x) = L(v)$ for any $v \in V$.

C. Let $P_3$ be the space consisting of all polynomial of degree no more than 3, and $D$ be the differentiation operator on $P_3$. Then, $\ker(D) = \{0\}$.

D. If $A$ and $B$ are similar matrices, then $\det(A - \lambda I) = \det(B - \lambda I)$ for any scalar $\lambda$. 
6. (6 points; 複選, 答對每個選項得 2 分, 答錯每個選項扣 2 分; 本題合計得分為負時,本題以 0 分計; 未作答亦以 0 分計) Which statements are correct?
A. Let \( u_1 = (-1,2,1) \), \( u_2 = (1,1,-2) \), \( v = (10,5,10) \), and \( S = \text{span}(u_1, u_2) \). The (shortest) distance between \( v \) and \( S \) is \( \frac{17\sqrt{30}}{7} \).

B. For the same setting in the previous question, the least square solution of the system \( x_1 u_1 + x_2 u_2 = v \) is \( x_1 = \frac{11}{7} \) and \( x_2 = -\frac{3}{7} \).

C. Let \( V \) be an inner product space, and \( \langle u_1, u_2 \rangle \) denote the inner product of any two vectors \( u_1, u_2 \in V \). If \( B = \{v_1, v_2, \ldots, v_n\} \) is an ordered basis of \( V \), then for any vector \( u \in V \), the coordinate of \( u \) can be given by \( [u]_B = \left[ \langle u, v_1 \rangle \langle u, v_2 \rangle \cdots \langle u, v_n \rangle \right]^T \).

7. (6 points; 複選, 答對每個選項得 2 分, 答錯每個選項扣 2 分; 本題合計得分為負時,本題以 0 分計; 未作答亦以 0 分計) Which statements are correct?
A. Assume \( A \) is a \( m \times n \) matrix and \( B \) is a \( m \times p \) matrix. If \( X \) is an \( n \times p \) unknown matrix, then the system \( A^TAX = A^TB \) always has a solution. (Here we assume \( m \geq n \).)

B. The matrix \( \begin{bmatrix} -\frac{1}{2} & \frac{i}{2} \\ \frac{i}{2} & \frac{3i}{2} \end{bmatrix} \) possesses a complete orthonormal set of eigenvectors.

\[ \begin{bmatrix} 2 & 0 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \]

C. \( \begin{bmatrix} 2 & 0 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \) is not defective.

8. (6 points; 複選, 答對每個選項得 2 分, 答錯每個選項扣 2 分; 本題合計得分為負時,本題以 0 分計; 未作答亦以 0 分計) Which statements are true?
A. Assume \( A_{3 \times 3} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \) and \( B_{3 \times 3} = \begin{bmatrix} 2a_2^T \\ a_1^T + a_2^T + a_3^T \end{bmatrix} \). If \( \det A = 2 \), then \( \det(AB^{-1}) = 1 \).

B. If \( P_{3 \times 3} \) is a projection matrix that projects any vector in \( \mathbb{R}^3 \) onto the vector
u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \text{ then there must be two eigenvectors that correspond to the eigenvalue of 0.}

C. If \( A \) is a 3\times3 matrix with 3 distinct eigenvalues 0, 1, 2, then the matrix \((A + I)\) must be invertible.

9. **(10 points)** Suppose there is an election every year in a country and the total population of this country remains fixed. If 60% of the people voted for K Party whereas 40% of the people voted for D Party in the election last time. However, 8% of K Party voters and 4% of D Party voters change their minds and vote for the rival party each year. What will the percentages of K Party and D Party voters be after \( n \) years, when \( n \) approaches infinity?

10. Consider the vector space \( C[0,1] \) with inner product defined by
\[
\langle f, g \rangle = \int_0^1 f(x)g(x)\,dx,
\]
where \( C[0,1] \) denotes the set of all real-valued functions that are defined and continuous on the closed interval [0,1].

A. **(5 points)** Use the Gram-Schmidt process to find an orthonormal basis for the subspace \( S \) spanned by 1, and \( x \).

B. **(5 points)** Find the best least squares approximation to \( e^x \) on the interval [0,1] by a function in \( S \).