**Instruction:** Read the following essay carefully, and then answer the following problems.


**Answer the following problems**

1. Rewrite this essay in Chinese around 100 words (15 %), give it a title and an abstract (15 %).

2. Answer the blanks (a) (b) (c) (d) as indicated in the essay. (20%)

3. Based on this essay, explain the terms induction, mathematical induction and any possible relationship between them. (15%)

4. Explain the roles and geometrical meanings played by figure a and figure b. (15%)

5. Following the procedure used in the essay, suggest a close form for $1^2 + 2^2 + 3^2 + \ldots + n^2$. (20%)

**Essay**

Induction is the process of discovering general laws by the observation and combination of particular instances. It is used in all sciences, even in mathematics. **Mathematical induction** is used in mathematics alone to prove theorems of a certain kind. It is rather unfortunate that the names are connected because there is very little logical connection between the two processes. There is, however some practical connection; we often use both methods together.

We are going to illustrate both methods by the same example. We may observe, by chance, that $1 + 8 + 27 + 64 = 100$ and recognize the cubes and the square, we may give to the fact we observed the more interesting forms: $1^2 + 2^2 + 3^2 + 8^2 = 10^2$. How does such a thing happen? Does it often happen that such a sum of success cubes is a square? In asking this we are like the naturalist who, impressed by a curious plant or a curious geological formation, conceives a general question. Our general question is considered with the sum of successive cubes (a). We were led to it by the particular instance $n = 4$.

What can we do for our question? What the naturalist would do; we can investigate other special cases. The special cases $n = 2, 3$ are still simpler, the case $n = 5$ is the next one. Let us add, for the sake of uniformity and completeness, the case $n = 1$. Arranging neatly all these cases, as a geologist would arrange his specimens of a certain ore, we obtain the following table:
\[1^3 = 1 = 1^2 \]
\[1^3 + 2^3 = 9 = 3^2 \]
\[1^3 + 2^3 + 3^3 = 36 = 6^2 \]
\[1^3 + 2^3 + 3^3 + 4^3 = 100 = 10^2 \]
\[1^3 + 2^3 + 3^3 + 4^3 + 5^3 = 225 = 15^2.\]

It is hard to believe that all these sums of consecutive cubes are squares by mere chance. In a similar case, the naturalist would have little doubt that the general law suggested by the special cases heretofore observed is correct; the general law is almost proved by induction. The mathematician expresses himself with more reserve although fundamentally, of course, he thinks in the same fashion. He would say that the following theorem is strongly suggested by induction:

*The sum of the first \( n \) cubes is a square.*

We have been led to conjecture a remarkable, somewhat mysterious law. Why should those sums of successive cubes be squares? But, apparently, they are squares. What would the naturalist do in such a situation? He would go on examining his conjecture. In so doing, he may follow various lines of investigation. The naturalist may accumulate further experimental evidence; if we wish to do the same, we have to test the next cases, \( n = 6, 7, 8, \ldots \). The naturalist may also examine the facts whose observation has led him to his conjecture; he compares them carefully, he tries to disentangle some deeper regularity, some further analogy. Let us follow this line of investigation.

Let us reexamine the case \( n = 1, 2, 3, 4, 5 \) which we arranged in our table. Why are all these sums squares? What can we say about these squares? Their bases are 1, 3, 6 10, 15. What would these bases? Is there some deeper regularity, some further analogy? At any rate, they do not seem to increase too irregularity. How do they increase? The difference between two successive terms of this sequence is itself increasing,

\[3 - 1 = 2, \ 6 - 3 = 3, \ 10 - 6 = 4, \ 15 - 10 = 5.\]

Now these differences are conspicuously regular. We may see here a surprising analogy between the bases of those squares, we may see a remarkable regularity in the numbers 1, 3, 6, 10, 15:
1 = 1
3 = 1 + 2
6 = 1 + 2 + 3
10 = 1 + 2 + 3 + 4
15 = 1 + 2 + 3 + 4 + 5.

If this regularity is general (and the contrary is hard to believe) the theorem we suspected take a more precise form:

\[ \text{It is, for } n = 1, 2, 3, 4, \ldots, \ 1^3 + 2^3 + 3^3 + \ldots + n^3 = (1 + 2 + 3 + \ldots + n)^2. \]

The law we just stated was found by induction, and the manner in which it was found conveys to us an idea about induction which is necessarily one-sided and imperfect but not distorted. Induction tries to find regularity and coherence behind the observations. Its most conspicuous instruments are generalization, specialization, analogy. Tentative generalization starts from an effort to understand the observed facts; it is based on analogy, and tested by further special cases. We refrain from further remarks on the subject of induction about which there is wide disagreement among philosophers. But it should be added that many mathematical results were found by induction first and prove later. Mathematics presented with rigor is a systematic deductive science but mathematics in the making is an experimental inductive science.

In mathematics as in the physical sciences we may use observation and induction to discover general laws. But there is a difference. In the physical science, there is no higher authority than observations and induction but in mathematics there is such an authority: rigorous proof. After having worked a while experimentally it may be good to change our point of view. Let us be strict. We have discovered an interesting result but the reasoning that led to it was merely plausible, experimental, provisional, heuristic; let us try to establish it definitely by a rigorous proof. We have arrived now at a “problem to prove”: to prove or to disprove the result stated before.

There is a minor simplification. We may know that \(1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}\). At any rate, this is easy to verify. Take a rectangle with sides \(n\) and \(n+1\), and divide it in two halves by a zigzag line as in figure a which shows the case \(n = 4\). Each of the halves is “staircase-shaped” and its area has the expression \(1 + 2 + 3 + \ldots + n\); for \(n = 4\), it is \(1 + 2 + 3 + 4 = 10\) as in figure b. Now, the whole area of the rectangle is \(n(n+1)\) of which the staircase-shaped area is one half; this proves the formula.
We may transform the result which we found by induction into

\[ 1^3 + 2^3 + 3^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2. \]

If we have no idea how to prove this result, we may at least test it. Let us test the first case we have not tested yet, the case \( n = 6 \). For this value, the formula yields

\[ 1 + 8 + 27 + 64 + 125 + 216 = \left(\frac{6 \times 7}{2}\right)^2 \]

and, on computation, this turns out to true, both sides being equal to 441.

We can test the formula more effectively. The formula is, very likely, generally true, true for all values of \( n \). Does it remain true when we pass from an value \( n \) to the next value \( n + 1 \)? Along with the formula as written above, we should also have

\[ 1^3 + 2^3 + 3^3 + \ldots + n^3 + (n+1)^3 = \left(\frac{(n+1)(n+2)}{2}\right)^2. \]

Now, there is a simple check. Subtracting from this the formula written above, we obtain

\[ (n+1)^3 = \left(\frac{(n+1)(n+2)}{2}\right)^2 - \left(\frac{n(n+1)}{2}\right)^2 \]

This is, however, easy to check. Our experimentally found formula passed a vital test. Let us see clearly what this test means. We verified beyond doubt that

\[ (n+1)^3 = \left(\frac{(n+1)(n+2)}{2}\right)^2 - \left(\frac{n(n+1)}{2}\right)^2. \]

We do not know yet whether \( 1^3 + 2^3 + 3^3 + \ldots + n^3 = \left(\frac{c}{6}\right) \) is true. But if we knew that this was true, we could infer, by adding the equation which we verified beyond doubt, that
$1^3 + 2^3 + 3^3 + \ldots + n^3 + (n+1)^3 = (d)$ is also true which is the same assertion for the next integer $n+1$. Now, we actually know that our conjecture is true for $1, 2, 3, 4, 5, 6$. By virtue of what we have just said, being true for $n = 6$, must also be true for $n = 7, \ldots$ and so on. It holds for all $n$, it is proved to be true generally.

The foregoing proof may serve as a pattern in many similar cases. What are the essential lines of this pattern? The assertion we have to prove must be given in advance, in precise form. The assertion must depend on an integer $n$. The assertion must be sufficiently "explicit" so that we have some possibility of testing whether it remains true in the passage from $n$ to the next integer $n+1$.

This process is so often used that it deserves a name. We would call it "proof from $n$ to $n+1" or still simpler "passage to the next integer”. Unfortunately, the accepted technical term is "mathematical induction”. This name results from a random circumstance. The precise assertion that we have to prove may come from any source, and it is immaterial from the logical viewpoint what the source is. Now, in many case, as in the case we discussed here in detail, the source is induction, the assertion is found experimentally, and so the proof appears as a mathematical complement to induction; this explains the name.

Here is another point, somewhat subtle, but important to anybody who desires to find proofs by himself. In the foregoing, we found two different assertions by observation and induction, one after the other, the first under $1$, the second under $2$; the second was more precise than the first. Dealing with the second assertion, we found a possibility of checking the passage from $n$ to $n+1$, and so we were able to find a proof by "mathematical induction”. Dealing with the first assertion, and ignoring the precision added to it by the second one, we should scarcely have been able to find such a proof. In fact, the first assertion is less precise, less “explicit”, less “tangible”, less accessible to testing and checking than the second one. Passing from the first to the second, from the less precise to the more precise statement, was an important preparative for the final proof.