1. **Knowledge Base**: A vector space is a set whose elements are called "vectors" and such that there are two operations defined on them: *i.e.*, you can add vectors to each other and you can multiply them by a scalar. These operations must obey certain axioms for a vector space. If a subset of a vector space is closed under addition and multiplication by scalars, then it is itself a vector space.

**Question (15 points)**: The damped harmonic oscillator is a model suited for many important physical problems: \( m \frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + kx = 0 \). Show that the set of solutions to this equation forms a vector space. Find a basis set for the space of solutions of the damped oscillator equation.

2. Consider the following coupled oscillators (20 points): The equations of motion for identical mass \( m_1 = m_2 \) and spring constant \( k_1 = k_2 = k \) can be described by

\[
m \frac{d^2x_1}{dt^2} = -kx_1 - k_3(x_1 - x_2) \quad \text{and} \quad m \frac{d^2x_2}{dt^2} = -kx_2 - k_3(x_2 - x_1).
\]

Find out the solution of the equations in terms of \((Ae^{i\omega t}, Be^{-i\omega t})\) by showing the eigenvalues and corresponding eigenvectors.

3. (1) Please solve (or integrate) the following differential equation to derive an algebraic expression that can be used to determine \( A(z) \) for \( z > 0 \) in terms of \( A(0) \). Here \( c_0 \) is a constant.

\[
\frac{d}{dz} A(z) = \frac{c_0}{1 + A(z)}
\]

(2) Please solve the following coupled differential equations to derive the algebraic expressions for \( A(z) \) and \( B(z) \) for \( 0 \leq z \leq L \), given that the boundary conditions are \( A(0) = 1 \) and \( B(L) = 0 \). Here \( \kappa \) is a real constant.

\[
\frac{d}{dz} A(z) = \kappa B(z)
\]

\[
\frac{d}{dz} B(z) = \kappa A(z)
\]

(3) Please solve the following coupled differential equations to derive the expressions for \( C_1(z) \) and \( C_2(z) \) for \( z > 0 \), given that the initial conditions are \( C_1(0) = C_2(0) = 0 \). Here \( \gamma \) and \( \kappa \) are real constants and \( f(z) \) is a real function of \( z \).

\[
\frac{d}{dz} C_1(z) = -\gamma C_1(z) + \kappa C_2(z)
\]

\[
\frac{d}{dz} C_2(z) = -\gamma C_2(z) + \kappa C_1(z) + f(z)
\]
4. (15%) (每小題 3 分)

(1) Please determine all the eigenvalues of the following 3x3 matrix A.

\[
A = \begin{bmatrix}
2 & -1 & 0 \\
-1 & 1 & -1 \\
0 & -1 & 2
\end{bmatrix}
\]

(2) Please determine all the eigenvectors of the above matrix A.

(3) Please determine a matrix U which can diagonalize A according to \( U^T A U = D \). Here D is a diagonal matrix with the eigenvalues of A as its diagonal elements and \( U^T \) is the transpose of U.

(4) Please derive the expression for the vector solution \( \bar{x} \) of the following linear equation:

\[
(A - \lambda I)\bar{x} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}
\]

Here \( \lambda \) is an arbitrary constant not equal to the eigenvalues of matrix A and I is the identity matrix. [Hint: Expansion in terms of eigenvectors]

(5) Please determine the lower triangular matrix L which can decompose the following 2x2 symmetric matrix B according to \( B = LL^T \).

\[
B = \begin{bmatrix} a & c \\ c & b \end{bmatrix} = LL^T = \begin{bmatrix} d & 0 \\ f & e \end{bmatrix}
\]

5. Expand \( f(x) = x^2 \), \( 0 < x < L \); (a) in a Cosine Series, (b) in a Sine Series, (c) in a Fourier Series. (15%)

6. A uniform slab of material with thermal diffusivity \( k \) occupies the space region \( 0 \leq x \leq L \) and initially has temperature \( U_0 \) throughout. Beginning at time \( t = 0 \), the face \( x = 0 \) is held at temperature zero; at the face \( x = L \), heat exchange takes place with a surrounding medium at temperature zero, so that \( hU(L,t) + U_x = 0 \) (where \( h \) is an appropriate heat transfer coefficient). We want to find the temperature \( U(x,t) \) of the slab at position \( x \) at time \( t \); \( U(x,t) \) satisfies the boundary value problem. (20%)