1. Given $P \in \mathbb{R}^{n \times n}$, $A \in \mathbb{R}^{n \times n}$ and $P$ is a positive definite matrix, find the rank of the matrix $(P - PA(A^T PA)^{-1} A^T P)$ (15%)

2. The following recurrence relation holds for the Hermite polynomial $H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$ where $n$ is degree. The first two polynomials are $H_0 = 1, H_1 = 2x$. Derive $H_n(x)$ (20%)

3. We call the quantity
   
   $$D_n(x) = \frac{\sin \left( \frac{n+1}{2} x \right)}{2 \sin \frac{x}{2}} = \frac{1}{2} + \sum_{k=1}^{n} \cos(k\alpha)$$

   the Dirichlet kernel.

   (1) Please prove this identity. (5%)
   (2) $D_n(x)$ is an even function of $x$. What is meant by an even function? (5%)
   (3) For every $n$, we have
   
   $$\frac{1}{n} \int_0^n D_n(x) dx = \frac{1}{2}$$

   Please prove it. (5%)

   (4) What is the period of $D_n(x)$? (5%)

4. A matrix $A$ is called normal if $A^T = AA^T$.

Suppose that $A = B + iC$, where $B$ and $C$ are real matrices. Show that $A$ is normal if and only if $BC = CB$. (10%)

5. Solve the following initial value problems:
   
   (a) $(D^2 + 4D + 5)y = 0 \quad y(0) = 0 \quad y'(0) = -3$ (10%)
   (b) $(D^2 - 2D + \pi^2 + 1)y = 0 \quad y(0) = 1 \quad y'(0) = 1 - \pi$ (10%)

6. Find solution $u(x, y)$ of the partial differential equation: (15%)
   
   $$u_x + u_y = 2(x + y)u$$