1. A fair coin is tossed 400 times. Let the random variable $X$ be the number of heads.
   (a) (4%) What is expected value of $X$?
   (b) (4%) What is the standard deviation of $X$?
   (c) (4%) Describe how you can estimate $P(100 \leq X \leq 200)$ by the central limit theorem.

2. Suppose we have a situation of rolling a die, where there is only one of six possible faces. We also suppose that all six outcomes have the same probability and the trials are independent.
   (a) (2%) Let the random variable $X$ be the number of rolls of a die until we see the first “4.” Determine the probability $P(X=k)$, $k \geq 1$.
   (b) (4%) Given the above probability distribution, what is the generating function? Show the result after simplification.
   (c) (4%) Use the generating function to find the mean of $X$.
   (d) (4%) Use the generating function to find the variance of $X$.
   (e) (8%) What is the expected number of rolls of a die until we will have seen all six faces?

3. (8%) There are 3 bags. One contains 2 red balls, another has 2 white balls and the third has one red ball and one white ball. You pick a bag at random, and without looking inside take out one ball. Suppose it is red. What is the probability that the other ball in that bag is also red? Show your reasoning.
4. (8%) Suppose we select independently both the $x$ and $y$ coordinates of a point in the $x$-$y$ plane from the same normal distribution which is centered at the origin

$$p(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

What is the expected distance of the random point in the $x$-$y$ plane from the origin?

5. (10%) Prove or disprove the following statement:

"In a vector space $V$, if $v_i$ and $v_j$ are linearly independent for $i, j = 1, 2, 3, \ i \neq j$, then $v_1, v_2, v_3$ are linearly independent."

6. Let $L$ be the linear operator that rotates vectors in $\mathbb{R}^2$ by 45° in the counterclockwise direction.

(a) (5%) Find the matrix representation of $L$ with respect to the natural basis $\{e_1, e_2\}$, where $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

(b) (5%) Find the matrix representation of $L$ with respect to the ordered basis $\{u_1, u_2\}$, where $u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $u_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
7. Let $u$ be an unit vector of $\mathbb{R}^n$ and $H = I - 2uu^T$. Please answer the following questions (MUST WITH REASON OR COUNTEREXAMPLE):

(a) (4%) Is $H$ a symmetric and orthogonal matrix?

(b) (4%) Is $H$ diagonalizable?

(c) (4%) Find $H^{-1}$ and $H^2$.

(d) (4%) Please find all the eigenvalues of $H$.

(e) (4%) Find the trace, rank and determinant of $H$.

(f) (6%) Find a matrix $H$, as stated above, such that $Hx = e_1$, where $x = (1/3, 2/3, 2/3)^T$ and $e_1 = (1, 0, 0)^T$.

(g) (4%) Is it possible to find a matrix $H$, as stated above, such that $Hx = e_1$, where $x = (-1, 3, -2)^T$ and $e_1 = (1, 0, 0)^T$?