In the following, \( \mathbb{R} \) denotes the set of all real numbers, \( n \) is any positive integer and \( \mathbb{R}^n \) is the Euclidean space containing all \( n \)-dimensional real column vectors. For \( 1 \leq i \leq n \), let \( e_i \) be a column vector whose transpose is equal to
\[
\begin{pmatrix}
0, \cdots, 0, 1, 0, \cdots, 0
\end{pmatrix}.
\]
The set \( \{e_1, \ldots, e_n\} \) is known as the canonical basis for \( \mathbb{R}^n \).

1. (12%) Which descriptions are correct and why? The solutions \( x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \) of
\[
Ax = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]
form a plane, line, point, subspace, null space of \( A \), column space of \( A \).

2. Let \( F \) be a vector space of dimension \( n \). Let \( T \) be a linear transformation on \( F \). We call the matrix \( B = (b_{i,j})_{1 \leq i,j \leq n} \) the matrix of \( T \) in the basis \( \{v_1, \ldots, v_n\} \) if
\[
Tv_j = \sum_{k=1}^{n} b_{k,j} v_k \quad \forall j = 1, 2, \ldots, n.
\]
Let \( T : \mathbb{R}^3 \to \mathbb{R}^3 \) be defined by
\[
T \left( \begin{bmatrix} a \\ b \\ c \end{bmatrix} \right) = \begin{bmatrix} 3c - b \\ 3b - a \\ a \end{bmatrix}.
\]

(a) (6%) Verify that \( T \) is a linear transformation.
(b) (7%) Find the matrix of \( T \) in the canonical basis of \( \mathbb{R}^3 \).

3. Let \( \{e_1, \ldots, e_n\} \) be the canonical basis of \( \mathbb{R}^n \) and \( \{w_1, \ldots, w_n\} \) be another basis of \( \mathbb{R}^n \). Let \( C \) be the matrix whose columns are \( w_1, \ldots, w_n \). The matrix \( C \) is then called the matrix of the change of basis from \( \{e_1, \ldots, e_n\} \) to \( \{w_1, \ldots, w_n\} \).

(a) (7%) Prove that the matrix \( C \) is invertible.
(b) (7%) Let \( T \) be a linear transformation of \( \mathbb{R}^n \) and \( B \) be the matrix of \( T \) in the canonical basis. Regard \( B \) as a linear transformation of \( \mathbb{R}^n \) under the matrix multiplication. Then the matrix \( A \) of \( T \) in another basis \( \{v_1, \ldots, v_n\} \) of \( \mathbb{R}^n \) satisfies \( A = D^{-1}BD \), where \( D \) is the matrix changing the basis from \( \{e_1, \ldots, e_n\} \) to \( \{v_1, \ldots, v_n\} \).

4. (a) (7%) Write the ellipsoid \( \frac{x_1^2}{4} + \frac{x_2^2}{9} + \frac{x_3^2}{16} = 1 \) in the form of \( x^tAx = 1 \), where \( x^t = (x_1, x_2, x_3) \) is the transpose of \( x \).
(b) (7%) State any three equivalent definitions of a real symmetric matrix \( A \) to be positive definite.
(c) (7%) Let \( x \in \mathbb{R}^3 \) and \( B \) be a \( 3 \times 3 \) matrix with real entries. Give conditions on \( B \) so that \( x^tBx = 1 \) represents as an ellipsoid in \( \mathbb{R}^3 \). Verify your claim.
5. Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear transformation and $A$ be the $n \times n$ matrix of $T$ in the canonical basis of $\mathbb{R}^n$. Assume that $T$ is a projection map, that is, $T^2 = T$.

(a) (5%) Prove that $A$ has 1 as an eigenvalue with multiplicity at least $\dim R(T)$, where $\dim V$ is the dimension of $V$ and $R(T)$ is the range of $T$.

(b) (5%) Show that the eigenvalue of $A$ is either 0 or 1.

(c) (5%) Prove that $A$ is similar to a diagonal matrix whose entries are either 0 or 1. (Two matrices $P, Q$ are said to be similar if there exists an invertible matrix $X$ such that $Q = X^T P X^{-1}$.)

(d) (5%) Show that the trace of $A$ ($\text{tr} A$) satisfies $\text{tr} A = \dim R(T)$.

(e) (5%) Prove that the similarity in (a) is orthogonal (that is, $X$ is an orthogonal matrix) if and only if $A$ is symmetric.

6. For any $n \times n$ matrix $K$, we let $K_{i,j}$ denote the $(i,j)$-th entry of $K$ for $1 \leq i, j \leq n$. In this setting, $K$ is called a stochastic matrix if

$$0 \leq K_{i,j} \leq 1, \quad \sum_{j=1}^{n} K_{i,j} = 1 \quad \forall 1 \leq i, j \leq n.$$

Assume in the following that $K$ is a stochastic matrix.

(a) (5%) Let $\rho(K)$ denote the spectrum of $K$, that is, the set of all eigenvalues of $K$. Prove that $|\lambda| \leq 1$ for all $\lambda \in \rho(K)$ and also $1 \in \rho(K)$.

(b) (5%) Show that if $v$ is an $n$-dimensional row vector such that $vK = v$, then $|v| K = |v|$, where $|v| = (|v_1|, \ldots, |v_n|)$ as $v = (v_1, \ldots, v_n)$.

(c) (5%) $K$ is said to be irreducible if, for any $1 \leq i, j \leq n$, there exists a positive integer $l = l_{i,j}$ such that the $(i,j)$-th entry of $K^l$, denoted by $(K^l)_{i,j}$, is positive, where $K^l$ is the matrix obtained by multiplying $K$ itself for $l$ times. Prove that if $K$ is irreducible, then 1 is a simple eigenvalue (that is, its multiplicity is one).