1. (13%) Assume $M_{22}$ denote the vector space consisting of all 2 by 2 matrices. Let $T: M_{22} \rightarrow M_{22}$ be a linear transformation defined by $T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+b & b+c \\ c+d & d \end{bmatrix}$ for any $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{22}$.

(a) (6%) Find the eigenvectors and eigenvalues of this transformation.

(Note that any vector in $M_{22}$ is actually a 2 by 2 matrix.)

(b) (7%) We choose $B = \{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \}$ as the basis of $M_{22}$. Both $\begin{bmatrix} a \\ c \end{bmatrix}$ and $T\begin{bmatrix} a \\ c \end{bmatrix}$ can be expressed as a linear combination of these four matrices. Assume we have $\begin{bmatrix} a \\ c \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ and $T\begin{bmatrix} a \\ c \end{bmatrix} = y_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + y_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + y_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + y_4 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Please find the matrix $A$ such that $[y_1, y_2, y_3, y_4]^T = A[x_1, x_2, x_3, x_4]^T$.

2. (12%) Assume $A = \begin{bmatrix} 1 & -1 & 1 & 4 & 1 \\ 2 & 1 & -1 & 5 & 5 \\ -1 & 2 & 1 & 1 & 3 \\ 1 & 1 & -2 & 0 & 2 \end{bmatrix}$

(a) (3%) What is the rank of this matrix?
(b) (3%) Find the nullspace of $A$.
(c) (3%) Find an orthonormal basis for the column space of $A$.
(d) (3%) Find the projection of $y = [1 \ 2 \ 4 \ 8]^T$ onto the column space of $A$.

3. (5%) A box holding 130 coins composed of some 1-dollar coins, 5-dollar coins and 10-dollar coins. The total value of those coins is 830 dollars. How many coins of each type are in the box? Note that your credits will be awarded proportionally to how well you apply the techniques of linear algebra to solve this problem.

4. (10%) Suppose that the probability of rain on tomorrow depends on today’s weather condition. Specifically, if today is a rainy day then the probability of rain tomorrow is 0.8; otherwise if today has no rain then the probability of rain tomorrow is 0.4. Now given that today’s weather is rainy, then what is the probability of having a rainy day on the 20th days from today? Solve the problem by using the theory of eigenvalues and eigenvectors. Write down and explain clearly all your derivation steps (including all the related eigenvalues, eigenvectors, matrices, ..., and etc.).
5. (10%) Consider an \( n \)-dimensional vector space over real field \( R \). Often, we use the standard basis \( \{e_i, i = 1, \cdots, n\} \) to span the vector space, where \( e_i \) is an \( n \)-element vector with the \( i \)th element being 1 and all the other elements being zero. Now given another two bases \( \{v_i, i = 1, \cdots, n\} \) and \( \{w_i, i = 1, \cdots, n\} \) for the vector space, where \( v_i \) is not orthogonal to \( v_j \), \( w_i \) is not orthogonal to \( w_j \), but \( v_i \) is orthogonal to \( w_j \), for \( i \neq j \). Given a vector \( x = [x_1, x_2, \cdots, x_n]^T \) in the space represented by the standard basis, explain clearly how you solve the weights \( \{a_i, b_i; i = 1, \cdots, n\} \) in an efficient way (without using matrix inversion or Gauss elimination operations) so that \( x \) can be represented as \( x = a_1 v_1 + \cdots + a_n v_n = b_1 w_1 + \cdots + b_n w_n \).

6. (10%) Find the eigenvalues and eigenfunctions of the boundary value problem
\[
y'' - 4\lambda y' + 4\lambda^2 y = 0
\]
\[
y(0) = 0, \quad y(1) + y'(1) = 0.
\]

7. (10%) Solve the initial value problem
\[
y'''' + y' = 2 + \sin x
\]
\[
y(0) = 0, \quad y'(0) = 1, \quad y''(0) = -1.
\]

8. (5%) Find the indicial equation of
\[
x^2 y'''' + x e^x y' + (x^3 - 1) y = 0
\]
if the solution is required near \( x = 0 \).

9. (5%) For the following initial value problem, find \( y_1(x) \) by the method of Laplace transforms.
\[
y_1' = 4y_1 - y_2,
\]
\[
y_2' = 2y_1 + y_2,
\]
\[
y_1(0) = 1, y_2(0) = 3
\]

10. (10%) Consider the periodic function \( f(t) \):
\[
f(t) = t, \quad 0 \leq t < 2\pi; \quad f(t + 2\pi) = f(t).
\]
(a) (2%) Express the Fourier series of \( f(t) \) in terms of the Fourier coefficients.
(b) (6%) Find the Fourier coefficients.
(c) (2%) At \( t = 0 \), the Fourier series converges to the value \( c \). Find \( c \).

11. (10%) Solve the initial-boundary value problem
\[
\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \quad (0 < x < \pi, \quad t > 0);
\]
\[
u(0, t) = u(\pi, t) = 0,
\]
\[
u(x, 0) = f(x) \quad (0 < x < \pi),
\]
\[
\frac{\partial u}{\partial t}(x, 0) = g(x) \quad (0 < x < \pi).
\]