1. (a) Let matrix $A = [a_1, a_2, a_3]$, where $a_1$, $a_2$, $a_3$ are vectors in $\mathbb{R}^3$. If $4a_1 - 3a_2 + 2a_3 = 0$, find $\det(A)$. 
   (2%) 

(b) Given the Adjoint of matrix $A$ denoted as $\text{adj}(A)$, find $\det(A)$, $A$, $\det(3A^{-1}A^T)$. 

Where 

\[
\text{adj}(A) = \begin{pmatrix} 2 & 1 & 0 \\ 4 & 3 & 2 \\ -2 & -1 & 2 \end{pmatrix}
\]

(6%) 

(c) For the linear system $Ax = b$ (6%) 

(i) Find the rank of $A$ and a basis for the column space of $A$. 

(ii) Find a basis for the null space $N(A)$? What is the dimension of $N(A)$? 

Where 

\[
A = \begin{pmatrix} 0 & 1 & 1 & 3 & 4 \\ 1 & -2 & 1 & 1 & 2 \\ 1 & 2 & 5 & 13 & 5 \\ -1 & 3 & 0 & 2 & -2 \end{pmatrix}
\]

2. For this problem, just give the answer, no need to show your computation. 

(a) Let $L$ be the linear transformation of the reflection about the line $ax + by = 0$, from $\mathbb{R}^2$ to $\mathbb{R}^2$, where $a^2 + b^2 \neq 0$. 

Find the matrix representation of $L$ with respect to the standard basis, find the dimension of the kernel space, and find a basis of the range space. (6%) 

(b) Let $L$ be the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ having the following matrix representation with respect to the standard basis; 

\[
\begin{bmatrix}
\cos \theta & 0 & \sin \theta \\
0 & 1 & 0 \\
-\sin \theta & 0 & \cos \theta
\end{bmatrix}
\]

Find $L(L(v))$, where 

\[
\begin{bmatrix} x \\ y \\ z \end{bmatrix}
\]

Find the dimension of the kernel space, and find a basis of the range space of $L$. (4%) 

(c) Let $A_1$ be the matrix representation of a linear transformation $L$ from $\mathbb{R}^n$ to $\mathbb{R}^n$ with respect to the basis $B_1$. Let $B_2$ be another basis of $\mathbb{R}^n$, and let $P$ be the transition matrix corresponding to the change of basis from $B_1$ to $B_2$. 

What is the matrix representation of $L$ with respect to the basis $B_2$? Express it in terms of $A_1$ and $P$. (2%)
3. Given an m by n matrix \( A = [a_{11}, a_{12}, \ldots, a_{mn}] \), where \( a_i \neq a_j \) if \( i \neq j \), \( 1 \leq i, j \leq n \).
   
   (a) What can you say regarding the properties of \( A^TA \)? (at least 3 statements to be made for full score of 4 points) (4%)
   
   (b) Let \( m=n=3 \) and \( f(x) = Ax \). Give two specific examples of \( A \) and explain the kinds of geometric operations thus involved respectively. (8%)

4. Let \( A = [a_{ij}] \) be an \( n \times n \) matrix with eigenvalues \( \lambda_1, \ldots, \lambda_n \). Show that
   
   \[ \sum_{j=1}^{n} (\lambda_j - a_{jj}) = 0. \] (6%)

5. Compute \( \cos(A) \) for \( A = \begin{bmatrix} -2 & -6 \\ 1 & 3 \end{bmatrix} \). (6%)

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6. Student A and student B are asked to solve the same initial value problem:
   \[ y'' + ay' + by = 0, \ y(0) = c, \ y'(0) = d. \] Unfortunately, student A mistook the constants \( a \) and \( c \) and got the solution \( y(t) = -2e^{-3t} - e^t \). Student B mistook the constants \( b \) and \( d \) and obtained the solution \( y(t) = e^t(-\cos 2t + \sin 2t) \). Assume we can trust these two students' capability of solving differential equations.
   
   i. (4%) What are the correct numbers of \( a, b, c, \) and \( d? \)
   
   ii. (10%) Solve the original initial value problem: \( y'' + ay' + by = 0, \ y(0) = c, \ y'(0) = d. \)

7. (12%) Solve \( x^3 \frac{d^3y}{dx^3} + 3x^2 \frac{d^2y}{dx^2} - 11x \frac{dy}{dx} + 16y = x^{-4}, \ x > 0.\)

8. (12%) Find a general solution of the system:
   \[ \begin{cases} x' = x + 2y + 3z \\ y' = y + 2z \\ z' = -2y + z \end{cases} \]

9. (12%) Determine the inverse Laplace transform of the function \( X(s) = \frac{2s^2 + 9s + 1}{(s+1)^2(s-2)}. \)