1. The point \( R = (x, x, x) \) is on a line through \((1, 1, 1)\). And, the point \( S = (y + 1, 2y, 1) \) is on another line.
   (a) (10%) Choose \( x \) and \( y \) to minimize the squared distance \( \| R - S \| ^2 \).
   (b) (5%) Find the minimum value of \( \| R - S \| ^2 \).

2. (a) Let

\[
A = \begin{pmatrix}
0 & 1 & 2 & 3 \\
2 & 6 & 6 & 3 \\
-1 & 0 & 0 & 3 \\
1 & 2 & -2 & 3
\end{pmatrix}
\]

(i) (3%) By applying row operations to find a lower triangular matrix \( L \) with 1’s on its diagonal and an upper triangular matrix \( U \) so that \( A = LU \).
(ii) (3%) Find the determinant of \( A \).

(b) (4%) Find the determinant of a real matrix \( K \), where

\[
K = \begin{pmatrix}
0 & a & b & c & d \\
-a & 0 & e & f & g \\
-b & -e & 0 & h & i \\
-c & -f & -h & 0 & j \\
-d & -g & -i & -j & 0
\end{pmatrix}
\]

3. Answer the following problems involving eigenvectors and/or linear transformation:

(a) (5%) Let the matrix \( A = \begin{pmatrix}
4 & 2 & 3 \\
-1 & 1 & -3 \\
2 & 4 & 9
\end{pmatrix} \). An eigenvalue of the matrix \( A \) is 3. Find a basis for the corresponding eigenspace.

(b) (5%) Compute \( A^8 \), where \( A = \begin{pmatrix}
4 & -3 \\
2 & -1
\end{pmatrix} \).
(c) (5%) A mapping $T: P_2 \rightarrow \mathbb{R}^3$ defined by

$$T(p) = \begin{bmatrix} p(-1) \\ p(0) \\ p(1) \end{bmatrix}$$

is a linear transformation. Here the vector space $P_2$ is the set of all polynomials of degree 2 or less; and $\mathbb{R}^3$ denotes the collection of all lists of three real numbers, usually written as $3 \times 1$ column matrices. Find the matrix for $T$ relative to the basis $\{1, x, x^2\}$ for $P_2$ and the standard basis for $\mathbb{R}^3$.

4. Answer the following problems dedicated to the orthogonality and its least-squares application:

(a) (5%) There are three vectors in real space $\mathbb{R}^3$:

$$u_1 = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad \text{and } y = \begin{bmatrix} -1 \\ -5 \\ 10 \end{bmatrix}.$$ 

Let $W$ be a subspace spanned by $u_1$ and $u_2$. Find a specific point in $W$, which is closest to $y$.

(b) (5%) Find a least-squares solution of the equation $Ax = b$ for

$$A = \begin{bmatrix} 1 & 5 \\ 3 & 1 \\ -2 & 4 \end{bmatrix}, \quad \text{and } b = \begin{bmatrix} 4 \\ -2 \\ -3 \end{bmatrix}.$$ 

5. Consider the differential equation $xy' + y = -2x^2y^2$, $x > 0$.

(a) (5%) Transform the above differential equation into a linear first-order differential equation.

(b) (5%) Find the general solution.
6. Consider the nonhomogeneous differential equation \( y'' + y = x \sin x \).

   (a) (5%) Find the solution of the associated homogeneous equation.
   (b) (8%) Find a particular solution.
   (c) (2%) Find the general solution.

7. (10%) Determine the Fourier series for the following periodical function:

\[
  f(x) = \begin{cases} 
  0, & -1 \leq x < 0 \\
  e^{-x}, & 0 \leq x \leq 1 
  \end{cases}, \quad f(x+2) = f(x)
\]

8. (15%) Solve the following problem through separation of variables and using the boundary conditions:

\[
\begin{align*}
  &u_{xx} = u_t, \quad 0 < x < 1, \quad t > 0 \\
  &u(0,t) = 0, \quad u(1,t) = 0, \quad t \geq 0 \\
  &u(x,0) = x(1-x), \quad u_t(x,0) = 0, \quad 0 \leq x < 1
\end{align*}
\]