1. (6%) Can the function below be autocorrelation function of a wide-sense stationary (WSS) random process? Justify your answer.

(a) \[ S_y(f) = H(f)H(-f)S_x(f) \]

(b) \[ Y(t) = \int_{-\infty}^{\infty} h(\tau)X(t-\tau)d\tau \]

(c) \[ S_y(f) = H(f)\bar{S}_x(f) \]

2. (a) (6%) Prove that \( S_y(f) = H(f)H(-f)S_x(f) \) if \( Y(t) = \int_{-\infty}^{\infty} h(\tau)X(t-\tau)d\tau \), where \( S_x(f) \) and \( S_y(f) \) are respectively the power spectral densities (PSDs) of the real-valued WSS signals \( X(t) \) and \( Y(t) \), and \( H(f) \) is the Fourier transform of the filter impulse response \( h(\tau) \).

(b) (4%) Show that the relation in (a) can be reduced to \( S_y(f) = |H(f)|^2 S_x(f) \), if \( h(\tau) \) is real.

(c) (4%) Use (b) to prove that the PSD of a real-valued WSS process is always non-negative.

3. In the figure below, \( \{a_n\} \) are unit impulses with amplitude \( \pm 1 \), whereas \( G(f) \), \( H(f) \) and \( C(f) \) are transfer functions corresponding to the impulse responses \( g(t) \), \( h(t) \) and \( c(t) \), respectively.

(a) (5%) In absence of noise \( w(t) \), namely, \( w(t) = 0 \), describe the Nyquist criterion for zero-ISI in the above baseband transmission system.

(b) (5%) Describe the model of the ideal Nyquist channel.

(c) (5%) Consider a rectangular pulse \( g(t) \), and a known channel impulse response \( h(t) \) as:
\[
g(t) = \begin{cases} 1, & 0 \leq t < T_b \\ 0, & \text{otherwise} \end{cases}
\]
and \( h(t) = \delta(t) + \delta(t - T_b) \).
where \( \delta(x) \) is the Dirac delta function. Find the matched filter impulse response \( c(t) \) that maximizes the signal-to-noise ratio at the output of the sampler in presence of the white noise \( w(t) \).

(d) (5%) Does \( c(t) \) in (c) satisfy the Nyquist Criterion? Justify your answer.

4. Consider a discrete memoryless source \( S \) with source alphabet \( S = \{ s_1, s_2, \ldots, s_K \} \) and occurrence probabilities \( \{ p_1, p_2, \ldots, p_K \} \).

(a) (10%) Denote the entropy of \( S \) as \( H(S) \). Find the values of \( p_1, p_2, \ldots, p_K \) so that \( H(S) \) is maximized. Prove your result.

(b) (10%) The second-order extension of this source is another discrete memoryless source \( T \) with source alphabet \( S^2 = \{ l_1, l_2, \ldots, l_M \} \), where \( M = K^2 \). Denote the occurrence probabilities of \( T \) as \( \{ q_1, q_2, \ldots, q_M \} \) and its entropy as \( H(T) \). Derive the relationship between \( H(S) \) and \( H(T) \).

5. Consider the (7,4) Hamming code defined by the generator polynomial \( g(X) = 1 + X + X^3 \).

(a) (4%) Find its parity-check polynomial \( h(X) \).

(b) (6%) If the received word is represented as \( r(X) = X + X^3 + X^5 \), determine

(i) the syndrome polynomial \( s(X) \) for this received word, and

(ii) the decoded message polynomial \( m(X) \).

6. Let \( \phi_1(t) = \cos w_1 t + \cos w_2 t, \quad \phi_2(t) = \cos w_1 t - \cos w_2 t, \quad w_1 = \frac{2\pi}{T}, w_2 = \frac{4\pi}{T} \) and \( T \) be the symbol duration.

(a) (5%) Are \( \phi_1(t) \) and \( \phi_2(t) \) orthogonal functions? Please verify your answer.

(b) (10%) If \( p_2(t) = a\phi_1(t) + b\phi_2(t) \) and \( p_2(t) = a\phi_1(t) - b\phi_2(t) \) are orthonormal basis functions, specify \((a, b)\) accordingly.

7. An FSK signal is given as:

\[
 s_0(t) = \frac{2E_b}{T} \cos w_0 t, \quad s_1(t) = \frac{2E_b}{T} \cos w_1 t
\]

where \( E_b \) is the bit energy, \( T \) is the bit duration and \( \{\cos w_0 t, \cos w_1 t\} \) is an orthogonal basis. The received signal \( x(t) = s_i(t) + w(t), \quad i = 0, 1, \) and \( w(t) \) is the added white gaussian noise with two-sided power spectral density \( N_0/2 \).

(a) (5%) Show the optimum receiver structure to detect \( x(t) \) and explain why it is optimum.

(b) (10%) Derive the corresponding bit error probability.