Notations.

1. The letter \( \mathbb{R} \) denotes the set of real numbers. Hence, the notation \( \mathbb{R}^n \)
   represents the usual Euclidean space of dimension \( n \).

2. The identity matrix of size \( n \) is denoted by \( I_n \).

3. For a matrix \( A \), we let \( A^t \) denote the transpose of \( A \), \( \text{tr} A \) the trace of
   \( A \), and \( |A| \) the determinant of \( A \). For a nonsingular square matrix \( B \),
   the notation \( B^{-1} \) means the inverse of \( B \).

4. For a given vector space \( V \), the notation \( \dim V \) denotes the dimension of \( V \). If \( S \) and \( T \)
   are subspaces of \( V \), then \( S + T \) denotes the subspace \( \{ u + v : u \in S, v \in T \} \).

5. If \( T \) be a linear transformation, then \( \text{Ker} T \) is the kernel of \( T \), while \( \text{Im} T \)
   is the image of \( T \).

6. The notation \( M_n(\mathbb{R}) \) represents the set of all \( n \times n \) matrices over \( \mathbb{R} \).

Problems.

1. (15 points.) Let \( \mathcal{U} \) be the solution space of
   \[ x_1 - x_2 + x_3 - x_4 = 0 \]
   in \( \mathbb{R}^4 \) and \( \mathcal{V} \) be the solution space of
   \[ x_1 - 2x_2 + x_4 = 0 \]
   \[ 2x_1 - x_2 + x_3 - x_4 = 0 \]
   \[ x_2 - x_3 - x_4 = 0 \]
   in \( \mathbb{R}^4 \). Is there a linear transformation \( T : \mathbb{R}^4 \to \mathbb{R}^4 \) so that \( Tu = u \)
   for all \( u \in \mathcal{U} \) and \( \text{Ker} T = \mathcal{V} \)? If so, represent \( T \) in matrix with respect to
   a basis of your choice of \( \mathbb{R}^4 \). Justify your answer.

2. (15 points.) Let
   \[ B = \begin{pmatrix} 1 & 3 & -3 \\ 0 & 4 & 5 \\ 0 & 0 & 9 \end{pmatrix}. \]
   Find all \( 3 \times 3 \) real matrices \( A \) such that \( A^2 = B \). Justify your answer.

3. (10 points.) Let \( A \) be a real \( 2 \times 2 \) matrix with positive entries. Prove or disprove
   that there is an eigenvector \( v \) of \( A \) such that its components are all positive.

4. (10 points.) Prove that for \( n \geq 2 \)
   \[ \begin{pmatrix} 1 & 1 & \ldots & 1 \\ x_1 & x_2 & \ldots & x_n \\ \vdots & \vdots & & \vdots \\ x_1^{n-2} & x_2^{n-2} & \ldots & x_n^{n-2} \\ x_1^n & x_2^n & \ldots & x_n^n \end{pmatrix} = \left( \sum_{j=1}^{n} x_j \right) \begin{pmatrix} 1 & 1 & \ldots & 1 \\ x_1 & x_2 & \ldots & x_n \\ \vdots & \vdots & & \vdots \\ x_1^{n-2} & x_2^{n-2} & \ldots & x_n^{n-2} \\ x_1^{n-1} & x_2^{n-1} & \ldots & x_n^{n-1} \end{pmatrix}. \]
5. Let \( \mathcal{V} \) be a vector space of finite dimension. Let \( S, T, \) and \( U \) be vector subspaces of \( \mathcal{V} \). Prove or disprove (by giving a counterexample) the following two formulas.

1. (10 points.) \( \dim(S + T) = \dim S + \dim T - \dim(S \cap T) \).
2. (10 points.) \( \dim(S + T + U) = \dim S + \dim T + \dim U - \dim(S \cap T) - \dim(T \cap U) - \dim(U \cap S) + \dim(S \cap T \cap U) \).

6. (1) (3 points.) Prove that any square matrix can be written as a sum of a symmetric matrix and a skew-symmetric matrix.

2. (6 points.) Let the linear transformation \( T : M_n(\mathbb{R}) \mapsto M_n(\mathbb{R}) \) be defined by \( T(A) = A^t \). Determine the eigenvalues and eigenspaces of \( T \).

3. (6 points.) Determine whether \( T \) is diagonalizable. If yes, diagonalize it; if not, prove it is not.

7. Let

\[
A = \begin{pmatrix} 3 & -2 & -2 \\ -2 & 0 & 1 \\ -2 & 1 & 0 \end{pmatrix}
\]

1. (5 points.) Find a matrix \( P \) such that \( P^{-1}AP \) is diagonal.

2. (5 points.) Find the maximum of \( X^tAX \) among all \( X \in \mathbb{R}^3 \) subject to \( X^tX = 1 \). Give an example of \( X \) that attains the maximum. Justify your answer.

3. (5 points.) Find the minimum of \( \text{tr}(Y^tAY) \) among all \( 3 \times 2 \) matrices \( Y \) subject to \( Y^tY = I_2 \). Give an example of \( Y \) that attains the minimum. Justify your answer.