Notations.

1. The letter $\mathbb{R}$ denotes the set of real numbers. Hence, the notation $\mathbb{R}^n$ represents the usual Euclidean space of dimension $n$.
2. The identity matrix of size $n$ is denoted by $I_n$.
3. For a matrix $A$, we let $A^t$ denote the transpose of $A$, $\text{tr} A$ the trace of $A$, and $|A|$ the determinant of $A$. For a nonsingular square matrix $B$, the notation $B^{-1}$ means the inverse of $B$.
4. For a given vector space $\mathcal{V}$, the notation $\dim \mathcal{V}$ denotes the dimension of $\mathcal{V}$. If $S$ and $T$ are subspaces of $\mathcal{V}$, then $S + T$ denotes the subspace \{ $u + v : u \in S, v \in T$ \}.
5. If $T$ be a linear transformation, then $\text{Ker} T$ is the kernel of $T$, while $\text{Im} T$ is the image of $T$.
6. The notation $M_n(\mathbb{R})$ represents the set of all $n \times n$ matrices over $\mathbb{R}$.

Problems.

1. (15 points.) Let $\mathcal{U}$ be the solution space of
\[ x_1 - x_2 + x_3 - x_4 = 0 \]
in $\mathbb{R}^4$ and $\mathcal{V}$ be the solution space of
\begin{align*}
    x_1 - 2x_2 + x_4 &= 0 \\
    2x_1 - x_2 + x_3 - x_4 &= 0 \\
    x_2 - x_3 - x_4 &= 0
\end{align*}
in $\mathbb{R}^4$. Is there a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ so that $Tu = u$ for all $u \in \mathcal{U}$ and $\text{Ker} T = \mathcal{V}$? If so, represent $T$ in matrix with respect to a basis of your choice for $\mathbb{R}^4$. Justify your answer.

2. (15 points.) Let
\[ B = \begin{pmatrix} 1 & 3 & -3 \\ 0 & 4 & 5 \\ 0 & 0 & 9 \end{pmatrix}. \]

Find all $3 \times 3$ real matrices $A$ such that $A^2 = B$. Justify your answer.

3. (10 points.) Let $A$ be a real $2 \times 2$ matrix with positive entries. Prove or disprove that there is an eigenvector $\mathbf{v}$ of $A$ such that its components are all positive.

4. (10 points.) Prove that for $n \geq 2$
\[
\begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-2} & x_2^{n-2} & \cdots & x_n^{n-2} \\ x_1^n & x_2^n & \cdots & x_n^n \end{vmatrix} = \left( \sum_{j=1}^{n} x_j \right) \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-2} & x_2^{n-2} & \cdots & x_n^{n-2} \\ x_1^n & x_2^n & \cdots & x_n^n \end{vmatrix}
\]
5. Let $\mathcal{V}$ be a vector space of finite dimension. Let $S$, $T$, and $U$ be vector subspaces of $\mathcal{V}$. Prove or disprove (by giving a counterexample) the following two formulas.

1. (10 points.) $\dim(S + T) = \dim S + \dim T - \dim(S \cap T)$.
2. (10 points.) $\dim(S + T + U) = \dim S + \dim T + \dim U - \dim(S \cap T) - \dim(T \cap U) - \dim(U \cap S) + \dim(S \cap T \cap U)$.

6. (1) (3 points.) Prove that any square matrix can be written as a sum of a symmetric matrix and a skew-symmetric matrix.

2. (6 points.) Let the linear transformation $T: M_n(\mathbb{R}) \mapsto M_n(\mathbb{R})$ be defined by $T(A) = A^t$. Determine the eigenvalues and eigenspaces of $T$.

3. (6 points.) Determine whether $T$ is diagonalizable. If yes, diagonalize it; if not, prove it is not.

7. Let

$$A = \begin{pmatrix} 3 & -2 & -2 \\ -2 & 0 & 1 \\ -2 & 1 & 0 \end{pmatrix}.$$ 

1. (5 points.) Find a matrix $P$ such that $P^{-1}AP$ is diagonal.

2. (5 points.) Find the maximum of $X^tAX$ among all $X \in \mathbb{R}^3$ subject to $X^tX = 1$. Give an example of $X$ that attains the maximum. Justify your answer.

3. (5 points.) Find the minimum of $\text{tr}(Y^tAY)$ among all $3 \times 2$ matrices $Y$ subject to $Y^tY = I_2$. Give an example of $Y$ that attains the minimum. Justify your answer.