1. (20%) Show that the moment of inertia of a rigid body $I_y$ is a symmetric matrix.

Prove also that the eigenvalues of any symmetric matrix must be real and
eigenvectors corresponding to distinct eigenvalues are orthogonal to each other.

2. (15%) For a function $f(z)$ of the complex variable $z$ with $f(z) = u(x, y) + iv(x, y)$,
show that the necessary conditions for $f(z)$ to be differentiable at
$z = z_0 = x_0 + iy_0$ are the Cauchy-Riemann conditions

$$
\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0), \quad \frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0)
$$

3. (15%) The function $f(x, y)$ has vanishing first derivatives at $(x_0, y_0)$, i.e.,
$\frac{\partial f}{\partial x}(x_0, y_0) = 0, \frac{\partial f}{\partial y}(x_0, y_0) = 0$. The second derivatives of this function at
$(x_0, y_0)$ are given by

$$
A = \frac{\partial^2 f}{\partial x^2}(x_0, y_0), \quad B = \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0), \quad C = \frac{\partial^2 f}{\partial y^2}(x_0, y_0).
$$

Let

us consider the quantity $\Delta = \det \begin{bmatrix} A & B \\ B & C \end{bmatrix}$. Please match the conditions on the left
to the consequences on the right by drawing lines (連線看):

**Conditions**

- $\Delta > 0$, and $A > 0$
- $\Delta < 0$
- $\Delta > 0$, and $A < 0$

**Consequences**

- $f$ has a saddle point at $(x_0, y_0)$
- $f$ has a local minimum at $(x_0, y_0)$
- $f$ has a local maximum at $(x_0, y_0)$

Justify your answers.
4. (25%) The lognormal distribution, which is useful in physics and statistics, may be defined as the distribution of a random variable $X$ whose logarithm is a Gaussian distribution.

The probability density function of $X$ can be given as

$$P(x) = \begin{cases} 
\frac{1}{\sqrt{2\pi} \sigma x} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right) & \text{for } x > 0 \\
0 & \text{for } x \leq 0,
\end{cases}$$

where $\mu$ and $\sigma^2$ are the mean and variance of the Gaussian distribution, respectively.

Find the mean $\mu_X = \langle X \rangle$ (10%) and variance $\sigma^2_X = \langle X^2 \rangle - \langle X \rangle^2$ (15%) of the lognormal distribution in terms of $\mu$ and $\sigma^2$, where the $n$-th moment of $X$ is defined as $\langle X^n \rangle = \int_{-\infty}^{\infty} x^n P(x) \, dx$.

5 (25%)

(a) (10%) Use the Laplace transform to solve the differential equation

$$\frac{d^2 y}{dt^2} + \omega^2 y = F \cos(\gamma t)$$

with the initial conditions $y(0) = 0$ and $y'(0) = 0$, if $\gamma \neq \omega$.

(b) (5%) If the difference between $\gamma$ and $\omega$ is very small, describe your solution.

(c) (5%) If $\gamma = \omega$, what is the solution?

(d) (5%) If a damping-force term is added into the differential equation in (a), what is the major change of the solution?