Notations:

1. $F = \mathbb{R}$ or $C$.

2. $F^{(n)}$ = the set of all column vectors $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$, where $a_i \in F$.

3. $M_n(F)$ = the set of all square matrices of dimension $n \times n$ with each elements in $F$.

1. (a) (5 points) State the definition of a basis $v_1, v_2, \ldots, v_n$ of $F^{(n)}$.

    (b) (5 points) Let $T$ be a linear transformation on $F^{(n)}$ and $v_1, v_2, \ldots, v_n$ be a basis for $F^{(n)}$. Define the matrix of $T$ in the basis $v_1, v_2, \ldots, v_n$.

    (c) (5 points) If you know the matrix $A$ of a linear transformation $T$ in the basis $v_1, v_2, \ldots, v_n$ of $F^{(n)}$. What is the matrix $B$ of $T$ in terms of $A$ in the basis $v_n, v_{n-1}, \ldots, v_1$ of $F^{(n)}$?

2. (a) (5 points) Let $A \in M_n(F)$. State the definition of the minimal polynomial of $A$.

    (b) (5 points) Prove that the minimal polynomial of $A$ is unique.

    (c) (5 points) Prove that every characteristic root (eigenvalue) of $A$ is a root of the minimal polynomial of $A$.

3. (a) (5 points) State the definition of a subspace of $F^{(n)}$.

    (b) (5 points) For $A \in M_n(F)$, let $V_a$ be the set $\{v \in F^{(n)} : (A - aI)^k v = 0 \text{ for some positive integer } k \text{ depending on } v\}$, where $a \in F$. Prove that $V_a$ is a subspace of $F^{(n)}$.

    (c) (5 points) Let $v \in V_a$ and $l$ be the first integer such that $(A - aI)^l v = 0$. Prove that $v, (A - aI)v, \ldots, (A - aI)^{l-1} v$ are linearly independent.

    (d) (5 points) If $a \neq b$ are in $F$, show that $V_a \cap V_b = \{0\}$. 
4. Determine whether the statement is true or false. If it is true, explain why. If it is false, give a counterexample.

(a) (5 points) If $A$ and $B$ are $n \times n$ real matrices and $B \neq O$, then $\det(A + xB) = 0$ for some $x$ in $R$.

(b) (5 points) If $A$ is an $n \times n$ real matrix, then the nullity of $A$ equals the nullity of the transpose $A'$ of $A$.

(c) (5 points) For any $n \times n$ real matrix $A$, $A'A = AA'$, where $A'$ is the transpose of $A$.

5. If $V$ is a finite dimensional vector space, $T : V \to V$ is a linear transformation such that $T^3 - 3T^2 + 3T - I = O$, where $O : V \to V$, $O(v) = 0$ for all $v \in V$.

(a) (5 points) Show that there is a $v \neq 0$ in $V$ such that $T(v) = v$.

(b) (5 points) Show that $T$ is invertible.

(c) (5 points) Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Show that $A$ is invertible, and express $A^{-1}$ as a polynomial in $A$ with real coefficients.

6. Let $A$ be an $m \times n$ real matrix, and let $A'$ be the transpose of $A$.

(a) (5 points) Show that if $m = 2$ and $n = 4$, then the determinant of $A'A$ is 0.

(b) (5 points) Write down certain conditions on $A$, $m$ and $n$ which will ensure that the determinant of $A'A$ is nonzero.

(c) (5 points) Show that if $v_1, v_2, \ldots, v_n$ are linearly independent vectors of $R^m$, then the determinant of

$$
\begin{bmatrix}
(v_1, v_1) & (v_1, v_2) & \cdots & (v_1, v_n) \\
(v_2, v_1) & (v_2, v_2) & \cdots & (v_2, v_n) \\
\cdots & \cdots & \cdots & \cdots \\
(v_n, v_1) & (v_n, v_2) & \cdots & (v_n, v_n)
\end{bmatrix}
$$

is positive, where $(,)$ is the standard inner product of $R^m$.

(d) (5 points) Is there any relationship between the rank of $A'A$ and the rank of $A$?