1. If an electric circuit contains a resistor $R$ (ohms) and a capacitor $C$ (farads) in series.

   The charge $q$ (coulombs) on the capacitor is given by

   $$ R \frac{dq}{dt} + \frac{q}{C} = E \quad E \text{ in volts.} $$

   If $i = \frac{dq}{dt}$, assume $i = 5$ amperes when $t = 0$, find $i$ ($R = 10$ ohms, $C = 10^{-3}$ farads, and $E(t) = 100 \sin 120\pi t$ volts). \hspace{1cm} (12\%)

2. Solve the system

   $$ \frac{dx}{dt} + x - y - z = t $$
   $$ \frac{dx}{dt} + 2 \frac{dy}{dt} + \frac{dz}{dt} - y = 0 $$
   $$ \frac{dx}{dt} - \frac{dy}{dt} - 4x + y = 6e^{2t} - 1 \quad (13\%) $$

3. Expand the function $f(x) = e^x$ in terms of eigenfunctions of the Sturm-Liouville problem

   $$ \frac{d^2y}{dx^2} + \lambda y = 0 $$

   with boundary conditions

   $$ y'(0) = 0 $$
   $$ y(\pi) = 0 \quad (15\%) $$

4. Find and discuss the general solution of the general second order Euler-Cauchy differential equation in detail. \hspace{1cm} (10\%)
5. Let \((x,y,z)\) represent the coordinates of a point in Euclidean space \(E_3\). Consider the operator defined by the following equations:

\[
\begin{align*}
    x' &= 2x + y + z \\
    y' &= -3x - y + 2z \\
    z' &= x - 3z
\end{align*}
\]

Show that the subspace defined by \(x + y + z = 0\) is invariant under this operator.

Find the representation of this operator in the subspace and find the eigenvalues of the representation. (12 %)

6. Let \(V\) be a finite dimensional space over \(C\), with a positive definite Hermitian form \(\langle , \rangle\).

Let \(A: V \rightarrow V\) be a linear map. Show that the following conditions are equivalent:

[A] We have \(AA^* = A^*A\).

[B] For all \(v \in V\), \(\|Av\| = \|A^*v\|\) (where \(\|v\| = \sqrt{\langle v, v \rangle}\)).

[C] We can write \(A = B + iC\), where \(B, C\) are Hermitian and \(BC = CB\).

(13 %)

7. Let the 3x3 matrix be defined as

\[
B = \begin{bmatrix}
1 & 0 & 0 \\
9 & -8 & 9 \\
6 & -6 & 7
\end{bmatrix}
\]

(a) Compute the eigenvalues of the matrix \(B\).

(b) Find a 3x3 matrix \(C\), a 3x3 matrix \(D\) and a real constant \(\alpha\) that satisfy the following equation: \(B^n = C + D\alpha^n\), \(n \in \mathbb{Z}, n \geq 1\). The integer \(n\) is greater than or equal to one. (10 %)
8. Let the $4 \times 5$ matrix be defined as

$$A = \begin{bmatrix} 1 & -2 & 3 & 0 & 0 \\ 3 & -7 & 9 & -2 & -3 \\ 1 & 3 & 3 & 4 & 9 \\ 1 & 1 & 3 & 7 & 10 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}$$

(a) Compute the rank of the matrix $A$. Hint: compute the reduced row echelon form of the matrix $A$.

(b) We can use the column vectors of the matrix $A$ to form a basis for the column space of the matrix $A$. The set of all column vectors is denoted as the set $S$, where $S = \{a_1, a_2, a_3, a_4, a_5\}$. Find a set $V \subset S$, where $V$ contains several vectors from the set $S$. This subset can form a basis for the column space of the matrix $A$. Hint: The selection can be deduced from the reduced row echelon form of the matrix $A$.

(c) Compute the dimension of the null space for the matrix $A$. The value is denoted as $\dim N(A)$ or the nullity of the matrix $A$. (15%)