離散數學

1. Find the smallest positive integer $x$ such that $13x = 1 \pmod{2436}$. (5%)

2. Prove that at a party where there are at least two people, there are two people who know the same number of other people there. (Assume that the acquaintance relation is a symmetric but non-reflexive relation.) (5%)

3. Suppose that a valid code is an $n$-digit number in decimal notation containing even number of 0's. Let $a_n$ denote the number of valid code words of length $n$. Find an explicit formula for $a_n$ (not recurrence relation). (5%)

4. Suppose that a connected planar simple graph with 300 edges. If a planar representation of this graph divides the plane into 200 regions, how many vertices does this graph have? (5%)

5. How many nonisomorphic unrooted trees are there with 5 vertices? (5%)

6. 填充題

1~2 格，每格 2 分；3~9 格，每格 3 分

請直接寫答案，計算過程不要寫在答案卷上

Consider the following graph

![Graph](attachment:graph.png)

Is it a planar graph? ___1___

Does it contain a Hamiltonian circuit? ___2___
Below is a spanning tree of the above graph.

Determine the fundamental cut-set corresponding to the edge \((a, b)\). \(3\)
Determine the fundamental circuit corresponding to the edge \((h, e)\). \(4\)

Let \(A = \{0, 1\}\) and \(B\) be the set of all functions from \(A^2\) to \(A\).
Define a binary relation \(<\) on \(B\) such that \(f < g\) if \(f(x, y) \leq g(x, y)\ \forall x, y\)
Then, \((B, <)\) is a lattice.
Then, \(\text{lub}(f, g) = 5\) and the universal lower bound is \(6\).

Mergesort takes \(7\) comparisons to sort 5 distinct elements in the worst case.
Any comparison-based sorting algorithm takes at least \(8\) comparisons to sort 5 distinct elements in the worst case.

Let \(A = \{1, 2, \ldots, 2048\}\) and \(R\) be a binary relation on \(A\) such that \(aRb\) if \(a\) divides \(b\).
Then, the set \(A\) can be partitioned into at least \(9\) antichains.
機率

1. What are the axioms of probability? (5%)

2. Let $X_1$ and $X_2$ be jointly continuous random variables with probability density function $f_{X_1,X_2}$. Let $Y_1 = X_1 + X_2$, $Y_2 = X_1 - X_2$. Find the joint density function of $Y_1$ and $Y_2$ in terms of $f_{X_1,X_2}$. (10%)

3. Suppose that it is known that the number of items produced in a factory during a week is a random variable with mean 75 and variance 25. What can be said about the probability that this week’s production will be between 60 and 90? (Hint: Use Chebyshev’s inequality.) (10%)

4. Suppose that $X$ takes on one of the values 0, 1, 2. If for some constant $c$,

$$P(X = i) = cP(X = i - 1), \quad i = 1, 2,$$

find $\text{Var}(X)$. (5%)

5. Let $X$, $Y$, and $Z$ be independent Poisson random variables with parameters $\lambda_1$, $\lambda_2$ and $\lambda_3$, respectively. For nonnegative integers $x$, $y$, and $z$, find

$$P(X = x, Y = y, Z = z | X + Y + Z = x + y + z).$$

(10%)

6. A standard Cauchy random variable has density function

$$f(x) = \frac{1}{\pi(1 + x^2)}, \quad -\infty < x < \infty$$

If $X$ is a standard Cauchy random variable, find the probability density function of $1/X$. (10%)