I. (10%) For what values of a and b does the improper integral
\[ \int_{0}^{\infty} \frac{x^{a}}{1+x^{b}} \, dx \] converge? Justify your answer.

II. (12%) Determine whether the following sequences of functions converge uniformly. Justify your answer.

(a) \( f_{n}(x) = \frac{1}{1 + n x} \) \( \forall x \in [0, 1] \), \( n \in \mathbb{N} \).

(b) \( f_{n}(x) = \frac{1}{1 + n x} \) \( \forall x \in [2, \infty) \), \( n \in \mathbb{N} \).

III. (12%) Let \( f(x) = \begin{cases} 0 & \text{if } x = \frac{1}{n}, \forall n \in \mathbb{N}, n \neq 1 \smallsetminus 1, \\ 1 & \text{otherwise}. \end{cases} \)

Riemann integrable on \([0, 1]\\)? Justify your answer.

IV. (14%) Let \( g(x, y) = \begin{cases} \frac{x y}{x^{2} + y^{2}} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases} \).

(a) Prove that \( g \) is continuous at \((0, 0)\).

(b) Is \( g \) differentiable at \((0, 0)\)? Justify your answer.

V. Suppose that \( f: \mathbb{R} \to \mathbb{R} \) is a non-zero differentiable function with the property that \( f(x+y) = f(x) \cdot f(y) \) \( \forall x, y \in \mathbb{R} \).

Prove that \( f(x) = e^{ax} \) for some \( a \in \mathbb{R} \).

VI. (12%) Let \( I \) be an interval containing \( x_0 \) and suppose that the function \( f: I \to \mathbb{R} \) is \( n \)-times differentiable and \( f^{(n)}(x) \) is continuous in \( I \). Show that if \( f^{(k)}(x_0) = 0, \forall k = 1, 2, 3 \) and \( f^{(n)}(x_0) \neq 0 \), then \( f(x_0) \) is a local extremum of \( f \).
VII. (10%) Suppose that the functions \( g : \mathbb{R} \to \mathbb{R} \) and \( h : \mathbb{R} \to \mathbb{R} \) have continuous 2nd-order derivatives. Define the function \( U : \mathbb{R}^2 \to \mathbb{R} \) by \( U(s, t) = g(s-t) + h(s+t) \) \( \forall s, t \in \mathbb{R} \). Prove that \( \frac{\partial^2 U}{\partial s^2}(s, t) - \frac{\partial^2 U}{\partial s^2}(s, t) = 0 \) \( \forall s, t \in \mathbb{R} \).

VIII. (20%) Let \( \mathcal{O} \) be an open subset in \( \mathbb{R}^3 \) and suppose that the functions \( f : \mathcal{O} \to \mathbb{R} \), \( g : \mathcal{O} \to \mathbb{R} \) are continuously differentiable. Define

\[ S = \{ (x, y, z) \in \mathcal{O} \mid g(x, y, z) = 0 \} \]

Suppose that \((x_0, y_0, z_0)\) in \( S \) is an extreme point of the function \( f : S \to \mathbb{R} \) and that \( \nabla g(x_0, y_0, z_0) \neq \mathbf{0} \). Use Implicit Function Theorem to prove that there exists a number \( \lambda \) such that

\[ \nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) \]