Discrete Mathematics

1. (10%) Let \( f(x) \) be a function of a real variable, and let \( \Delta f \) be the function \( \Delta f(x) = f(x+1) - f(x) \). For \( n > 1 \), define \( \Delta^n f = \Delta(\Delta^{n-1} f) \). Show that \( \Delta^n f(x) = \sum_{i=0}^{n} (-1)^i C(n,i) f(x+n-i) \) for \( n > 0 \).

   where \( C(n,i) = \frac{n!}{i!(n-i)!} \).

2. (10%) Let \( A \) be a nonempty set and let \( P \) be the power set of \( A \). Define the following relations:

   \( R_1 = \{(X, Y) \in P \times P: X = Y\} \),
   \( R_2 = \{(X, Y) \in P \times P: X \subseteq Y\} \),
   \( R_3 = \{(X, Y) \in P \times P: X \subseteq Y\} \),
   \( R_4 = \{(X, Y) \in P \times P: X \neq Y\} \),
   \( R_5 = \{(X, Y) \in P \times P: X \cap Y = \emptyset\} \), and
   \( R_6 = \{(X, Y) \in P \times P: X \cap Y = \emptyset\} \).

(a). List the equivalence relations.
(b). List the partial ordering relations.
(c). List the strict ordering relations.

3. (10%) Let \( S = \{0, 1, 2, \ldots, 2^{16} - 1\} \) and \( T = (S - \{0\}) \cup \{2^{16}\} \). Let \( \oplus \) be a binary operation on \( T \) defined by \( x \oplus y = x \cdot y \mod (2^{16} + 1) \) for \( x, y \in T \). (Note that \( 2^{16} + 1 \) is a prime number.)

(a). Prove or disprove that \((T, \oplus)\) is an abelian group.
(b). Let \( f \) be a mapping from \( T \) to \( S \) defined by \( f(2^{16}) = 0 \) and \( f(x) = x \) for \( x \in T - \{2^{16}\} \).

   Is it possible that \( f \) is an isomorphism from \((T, \oplus)\) to \((S, \oplus)\) for some binary operation \( \oplus \) on \( S \)?

4. (10%) An \( \epsilon \)-permutation of \( 2n \) objects is defined as the following statements:

   "The \( 2n \) objects are rearranged into two rows, \( n \) objects in each row, so that the number of permutations in one row is equal to the number of permutations in another row."

   Assume that there are 10 red balls, 5 black balls, and 5 white balls. Balls of the same color are considered identical. Find the number of \( \epsilon \)-permutations of the above 20 balls.

5. (10%) An undirected graph \( G = (V, E) \) is called a bipartite graph if the set of vertices \( V \) can be partitioned into two nonempty subsets \( X \) and \( Y \) such that there is no edge in \( E \) joining two vertices in \( X \) or in \( Y \). A bipartite graph is complete if there is exactly one edge between \( x \) and \( y \), \( \forall \) vertices \( x \in X \) and \( y \in Y \). Let \( S_Y = \{ G = (V, E) | G \text{ is a complete bipartite graph} \} \) and \( F((V, E)) = |E| \), where \( |E| \) is the number of elements in \( E \). Find

(a). \( |S_Y| \)
(b). \( F(S_Y) \), where \( F(S_Y) = \sum_{X \in S_Y} F(x) \)
1. (10%) Let 
\[
A = \begin{bmatrix}
1 & 2 & 4 & 1 \\
1 & 1 & 3 & 2 \\
2 & 3 & 7 & 3 \\
4 & 5 & 13 & 7
\end{bmatrix}
\]
(a) (2%) Find the dimension of and a basis for the nullspace of A.
(b) (1%) Find the dimension of and a basis for the row space of A.
(c) (2%) Find the dimension of and a basis for the column space of A.
(d) (2%) If B is a nonsingular 3 \times 3 matrix, what is the dimension of the nullspace of BA? Why?
(e) (3%) Let \( b = (8, 3, 11, 17)^T \). Is \( b \) in the column space of \( A \)? How many solutions will the linear system \( Ax = b \) have?

2. (7%) The sets \( E = [1 + t, t + t^2, 1 + t^2] \) and \( F = [1, 1 + t, 1 + t + t^2] \) are both ordered bases for the vector space \( P_3 \) (\( P_3 \) denote the set of polynomials of degree less than 3).
(a) (2%) What is the coordinates of \( 7 + 5t + 9t^2 \) with respect to the basis \( F \)?
(b) (5%) Find the transition matrix from \( E \) to \( F \).

3. (8%) Let 
\[
A = \begin{bmatrix}
1 & 2 \\
1 & 3
\end{bmatrix}, \quad
B = \begin{bmatrix}
1 & 2 \\
2 & 4
\end{bmatrix}, \quad
C = \begin{bmatrix}
-1 & -2 \\
-3 & -5
\end{bmatrix}, \quad
D = \begin{bmatrix}
-1 & -2 \\
0 & -2
\end{bmatrix}
\]
(a) (4%) Test matrices \( A, B, C \), and \( D \) for linear independence or dependence.
(b) (4%) Find the dimension of and a basis for the space spanned by \( A, B, C \), and \( D \).

4. (8%) Given the five data points: \((x,y) = (-2,3), (-1, 5), (0, 5), (1, 4), \) and \((2, 3)\), find the linear function \( y = \beta_0 + \beta_1x \) which best fits the data in the least square sense. In the measurement, the last three data points are less reliable. The fitting errors of the three points are supposed to be weighted half as much as the first two data points in evaluating the least square errors.

5. (8%) Define the transformation from \( R^2 \) to \( R^2 \) by \( T(x) = Ax \), where
\[
A = \begin{bmatrix}
7 & 2 \\
-4 & 1
\end{bmatrix}
\]
Suppose \( B \) is the basis for \( R^2 \) with respect to which the transformation \( A \) is a diagonal matrix. Find the diagonal matrix and the basis \( B \).

6. (9%) Let \( T : P_2 \rightarrow P_3 \) be the transformation that maps a polynomial \( c_0 + c_1t \) into the polynomial \( c_1 + (c_0 + c_1)t + (c_0 - c_1)t^2 \). Find the matrix for \( T \) with respect to the ordered bases \( \{1 + 2t, 3 + t\} \) and \( \{1, 1 + t, 1 + t + t^2\} \).